# Mobile robot control Part 2 : control of chained systems and application to path following and time-varying point-stabilization of wheeled vehicles

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# MOBILE ROBOT CONTROL, PART 2: CONTROL OF CHAINED SYSTEMS AND APPLICATION TO PATH FOLLOWING AND TIME-VARYING POINT-STABILIZATION OF WHEELED VEHICLES

COMMANDE DE ROBOTS MOBILES, DEUXIEME PARTIE: COMMANDE DES SYSTEMES CHAINES ET APPLICATION AU SUIVI DE TRAJECTOIRES ET A LA STABILISATION INSTATIONNAIRE DE VEHICULES A ROUES

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## Abstract

The first part of the study is centered on control design and analysis for nonlinear systems which can be converted to *chained-form* systems. Solutions to various control problems (open-loop steering, partial or complete state feedback stabilization) are either recalled, generalized, or developped by extending an approach followed by the author in previous papers on mobile robot control. In particular, globally stabilizing *time-varying* feedbacks are derived and a discussion of their convergence properties is provided.

Application to the control of nonholonomic wheeled mobile robots is described in the second part of the study, by considering the case of a car pulling trailers. This application encompasses simpler unicycle-type and car-like vehicles without trailers. Finally it is shown how slightly modified chained systems can be introduced to derive controls with a broader domain of stability.



**Key words:** mobile robots, nonholonomy, chained systems, feedback stabilization, time-varying feedbacks.



## Résumé

La première partie de l'étude est axée sur la commande des systèmes non-linéaires dont les équations peuvent se mettre sous la forme dite *chaînée*. Des solutions de différents problèmes (commande en boucle ouverte pour transférer le système entre deux configurations quelconques, stabilisation partielle ou complète du système par retour d'état) sont soit retrouvées, généralisées, ou bien développées en étendant une approche considérée par l'auteur dans de précédents articles sur la commande de robots mobiles. En particulier, des commandes par retour d'état *instationnaire* globalement stabilisantes sont proposées, et une analyse préliminaire de leurs propriétés de convergence est réalisée.

L'application à la commande de robots mobiles à roues non-holonomes est décrite dans la deuxième partie de l'étude, en considérant le cas d'une voiture (ou d'un camion) à laquelle sont attelées des remorques. Cette application contient les cas plus simples de véhicules de type unicycle ou voiture sans remorques. Il est finalement montré comment l'introduction de formes chaînées légèrement modifiées permet de synthétiser des commandes avec un domaine de stabilité plus étendu.



**Mots clés:** robots mobiles, non-holonomie, systèmes chaînés, stabilisation par retour d'état, retours d'état instationnaires.

# Contents

**1 Introduction** **4**

**2 Control of chained-form systems** **5**
- 2.1 About chained systems . . . . . . . . . . . . . . . . . . . . . . . . . 5
- 2.2 Skew-symmetric chained form and Lyapunov control design . . . 8
- 2.3 Continuous time-varying feedbacks and exponential convergence in the 3-dimensional case . . . . . . . . . . . . . . . . . . . . . . . 15
- 2.4 Asymptotical rate of convergence . . . . . . . . . . . . . . . . . . . 20

**3 Application to the control of a car with -n- trailers** **25**
- 3.1 Modelling equations and notations . . . . . . . . . . . . . . . . . . 25
- 3.2 About path planning . . . . . . . . . . . . . . . . . . . . . . . . . . 29
- 3.3 Smooth feedback controls for path following and stabilization about a fixed desired configuration . . . . . . . . . . . . . . . . . . 30
- 3.4 A modification to broaden the stability domain . . . . . . . . . . . 33

# 1 Introduction

From a theorem due to Brockett [5], it is now well known that nonholonomic wheeled mobile robots with restricted mobility (such as unicycle-type and car-like vehicles) cannot be stabilized about a desired configuration (or posture) via continuous pure-state feedback [3], [23], [1]. Non-smooth feedback has been proposed as an alternative solution (see [4], [6], [29], for example). Another alternative, first pointed out by the author in [24], consists of using smooth *time-varying* feedbacks, i.e. feedbacks which explicitly depend on the time variable. Such feedbacks had previously been very little studied in Control Theory. The result given in [24], for a unicycle-type vehicle, has subsequently motivated research work in order to explore the potentialities of time-varying feedbacks [8], [9], [11], derive explicit design methods [18], [31], and extend their use in robotics applications [25], [27].

The possibility of modelling the kinematic equations of wheeled mobile robots by so-called canonical *chain form* equations (a particular class of nonlinear nilpotent systems) has been pointed out in [16] when treating the case of car-like vehicles. It was known before that the equations of unicycle-type vehicles (a simpler case) could be written in this form, but this had not been used explicitely at the control design level. More recently, it has been shown [28] that the equations of vehicles with trailers could also be locally converted into a chained form.

In [16], the authors aimed essentially at exploring methods for open-loop steering of nonholonomic systems by using sinusoidal inputs. More recently, the authors of [31] have realized that chained systems could also be put under another canonical form, called "power form", and that power form systems belonged to the class of systems considered in [18] for which explicit smooth time-varying stabilizing feedbacks can be derived. The method proposed in [31], for deriving such controls, is applied to the problem of locally stabilizing a car-like system about a desired posture. A global solution to this problem had previously been given in [25] by using another approach.

The Section 2 of the present study focuses on the control of chained form systems. After pointed out some facts about these sytems and recalling some results about the open-loop steering problem, it is shown that chained form systems can themselves be converted into a slightly different form, named here *skew-symmetric chained form*, particularly well adapted to subsequent Lyapunov design and analysis of globally stabilizing time-varying feedbacks. In the process of deriving such control laws, useful connections with more classical linear control techniques are explicited. In the three dimensional case, a set of exponentially converging continuous time-varying feedbacks is also derived. Existing solutions to the point-stabilization problem are then tentatively compared by analysing the type of stability associated with each of them.

In Section 3, the results of Section 2 are applied to a car pulling -n- trailers, seen as an extension of the unicycle and car cases. The same approach as in [27], according to which stabilization about a desired configuration can be treated as an extension of the path following problem, is considered. Finally, a slight modification in the modelling of the system's equations is proposed so as to broaden the control stability domain.

# 2 Control of chained-form systems

## 2.1 About chained systems

Let us thus consider a chained form system which may be written as follows:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_1 x_3 \\
\dot{x}_3 &= u_1 x_4 \\
&\vdots \\
\dot{x}_{n-1} &= u_1 x_n \\
\dot{x}_n &= u_2
\end{aligned} \tag{1}$$

or, equivalently:

$$\dot{X} = h_1(X)u_1 + h_2 u_2 \qquad h_1(X) = \begin{bmatrix} 1 \\ x_3 \\ x_4 \\ \vdots \\ x_n \\ 0 \end{bmatrix} \qquad h_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

With respect to the notations used in [16], the components of the state vector have just been ordered differently.

In the second part of the study, where the system 1 will be used to model the kinematic equations of wheeled mobile robots, $x_1$ **will represent the distance gone by the vehicle along a path to be followed**, and $x_2$ **will represent the lateral distance between the vehicle and the path**. Path following will thus mainly consists in regulating $x_2$ about zero, independently of the values taken by $x_1$ (and thus $u_1$), while stabilization about a desired configuration will further involve the regulation of $x_1$ about zero by utilizing also the input $u_1$.

It is worth noting that a chained system like 1, although it is nonlinear, has a strong underlying linear structure. This clearly appears when $u_1$ is taken as a function of time and no longer as a control variable. In this case, the system becomes a single-input time-varying linear system which may be written as follows:

$$
\begin{aligned}
\dot{\tilde{x_1}} &= 0 \\
\dot{X_2} &= \begin{bmatrix} 0 & u_1(t) & 0 & \cdots & \cdots & 0 \\ 0 & 0 & u_1(t) & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & u_1(t) & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & u_1(t) \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} X_2 + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2
\end{aligned} \tag{3}
$$

with:
$\tilde{x_1} = x_1 - \int_0^t u_1(\tau)d\tau$, and $X_2^T = [x_2, x_3, \ldots, x_n]$.

Putting a two-input nonlinear system in the chained form, when it is possible, is thus equivalent to linearizing this system with respect to one of its inputs. Since the chained system is controllable, controllability of the original system is a necessary condition. Sufficient conditions are given in [17].

When the input $u_1$ is taken as a function of time, the system is clearly no longer controllable, due to the first equation. However, under certain conditions upon the choice of $u_1(t)$, the second part of the system involving $X_2$ remains controllable.

This type of property is very useful. For instance, it is used further in the study for the derivation of smooth time-varying feedbacks which asymptotically stabilize the point $X = 0$. It can also be utilized to solve the **open-loop steering problem**, i.e. the problem of determining open-loop control inputs that steer the system to a desired configuration $X_{desired}$, chosen equal to zero without loss of generality. The method basically consists of two steps: i) choose an integrable function $u_1(t)$ which ensures controllability of the second part of the system, and determine a control $u_2(t)$ which drives $X_2(t)$ to zero in finite time (usually done by integrating the system's equations on some time interval and solving a set of algebraic equations), and ii) once $X_2$ is at zero, keep $u_2$ equal to zero so as to leave $X_2$ unchanged, and determine $u_1(t)$ so as to drive $x_1(t)$ to zero in finite time.

This method is used in [15], with $u_1(t)$ and $u_2(t)$ chosen as piecewise constant inputs. In this case, the first step corresponds to discretizing the system's equations with $u_1$ being kept constant (and different from zero) over $n-1$ sampling time intervals $\Delta$, and applying a dead-beat control strategy (the poles of the controlled discretized system are set equal to zero) to determine the values of $u_2$ on the time intervals $[k\Delta, (k+1)\Delta[\ (0 \leq k \leq n-2)$. At time $t = (n-1)\Delta$, $X_2$ has reached zero and $u_1$ may then be chosen equal to $-x_1((n-1)\Delta)/Delta$ so as to have $x_1$ equal to zero at time $t = n\Delta$. Note that, by working some more on the choice of $u_1$, feedback versions of this technique can be obtained. It can also be shown that multiplying the piecewise constant inputs by $(1 - cos(\omega t))$, with $\omega = \frac{2\pi}{\Delta}$, does not change the values of $X$ at the sampling instants. This a simple way of changing piecewise constant inputs into continuous inputs that achieve the same result.

A variant of this method is considered in [16], with $u_1(t)$ and $u_2(t)$ being composed of sinusoids at integrated related frequencies. Obviously, the same method applies with other inputs. A more geometrical method for open-loop steering of nonholonomic vehicles will also be pointed out further in Section 3.2.

When $u_1$ is **constant and different from zero**, the above system becomes time-invariant and the second part of the system is clearly controllable. By applying classical linear control techniques, it is then possible to derive linear feedbacks $u_2(X_2)$ which stabilize the origin $X_2 = 0$ exponentially.

In fact, even if $u_1(t)$ is not constant but only **piecewise continuous, bounded, and strictly positive (or negative)** it is quite simple to derive stabilizing feedbacks $u_2(X_2)$ for the second part of the system. Indeed, since $x_1(t)$ varies monotically with time, differentiation with respect to time can be replaced by differentiation with respect to $x_1$. From now on we will refer to this change of variable as the $u_1$**-time- scaling** procedure. Then, the second part of the system may equivalently be written:

$$\begin{array}{rcl} x_2^{(1)} & = & sign(u_1)x_3 \\ x_3^{(1)} & = & sign(u_1)x_4 \\ & \vdots & \\ x_{n-1}^{(1)} & = & sign(u_1)x_n \\ x_n^{(1)} & = & sign(u_1)v_2 \end{array} \tag{4}$$

with:
$x_i^{(j)} = sign(u_1)\frac{\partial^j x_i}{\partial x_1^j}$, and $v_2 = u_2/u_1(t)$.

This is the equation of a linear invariant system, an equivalent input-output representation of which is:

$$x_2^{(n-1)} \;\; = \;\; sign(u_1)^{n-1}v_2 \tag{5}$$

One falls upon a controllable invariant linear system which admits exponentially stable linear feedbacks in the form:

$$v_2(X_2) = -sign(u_1)^{n-1}\sum_{i=1}^{i=n-1} k_i x_2^{(i-1)} \qquad (k_i > 0, \; \forall i) \tag{6}$$

Hence, the time-varying control:

$$u_2(X_2, t) = u_1(t)v_2(X_2) \tag{7}$$

globally asymptotically stabilizes the origin $X_2 = 0$ in this case. Moreover, the trajectories followed by the system's solutions are invariant with respect to variations of $u_1(t)$.

This "feedback linearization" technique, associated with $u_1$-*time-scaling*, has in fact been used by other authors working on mobile robot control. For example, Sampei *et al.* [20] have applied it to the problem of following a straight line in the case of a car pulling a single trailer. Their solution however differs from the one given further in the article in that they took the car's steering wheel angle as a control, instead of the angle's velocity.

In the earlier work of Dickmanns and Zapp [10], on vision-based road-line following, $u_1$-*time-scaling* is also implicitly used together with tangent linearization of the system's equations, instead of exact feedback linearization. In their work, $u_1$ has the physical meaning of the car's translational velocity.

Extension of the path following problem to the point-stabilization problem in order to achieve smooth time-varying feedback stabilization of a unicycle-type vehicle about a given posture, based on $u_1$-*time-scaling*, has been first proposed in [27]. The present study may be seen as a generalisation of the results described in this paper.

## 2.2 Skew-symmetric chained form and Lyapunov control design

We show next, by introducing the skew-symmetric chained form evoked before, and via a Lyapunov-like analysis, that the control 7 globally stabilizes the origin $X_2 = 0$ of the second part of the chained system, provided that $|u_1(t)|$ and $|\dot{u}_1(t)|$ are bounded, and $u_1(t)$ **does not asymptotically tend to zero**. An important difference with the result stated previously is that $u_1(t)$ **is now allowed to pass through zero**.

From there, it will be simple to complement the analysis and derive smooth time-varying feedbacks which globally stabilize the origin $X = 0$ of the complete system.

To this purpose, let us consider the following change of coordinates $\phi_1 : X \longmapsto Z$ in $R^n$:

$$\begin{array}{rcl} z_1 & = & x_1 \\ z_2 & = & x_2 \\ z_3 & = & x_3 \\ & \vdots & \\ z_{j+3} & = & k_j z_{j+1} + L_{h_1} z_{j+2} \qquad 1 \leq j \leq n-3 \end{array} \tag{8}$$

where:

- $k_j$ $(1 \leq j \leq n-3)$ is a real positive number;
- $L_{h_1} z_j = \frac{\partial z_j}{\partial X} h_1(X)$: the Lie derivative of $z_j$ along $h_1$;

- $L_{h_1}^k = L_{h_1}^{k-1} L_{h_1}$: the Lie differentiation operator of order $k$ along $h_1$.

One easily verifies that the Jacobian matrix $\frac{\partial \phi_1}{\partial X}$ is a constant lower triangular matrix with ones on the diagonal. It is thus a regular linear change of coordinates in $R^n$.

Moreover: $L_{h_2} z_i = 0$ $(1 \leq i \leq n-1)$, and $L_{h_2} z_n = 1$.

Taking the time derivative of $z_{j+3}$ and using 2:

$$\begin{array}{rcl} \dot{z}_{j+3} & = & \frac{\partial z_{j+3}}{\partial X} \dot{X} \\ & = & (L_{h_1} z_{j+3}) u_1 \; + \; (L_{h_2} z_{j+3}) u_2 \end{array} \tag{9}$$

Also,from 8:

$$L_{h_1} z_{j+3} = -k_{j+1} z_{j+2} + z_{j+4} \tag{10}$$

Hence:

$$\dot{z}_{j+3} \; = \; -k_{j+1} z_{j+2} \; + \; z_{j+4} \qquad (0 \leq j \leq n-4) \tag{11}$$

and:

$$\dot{z}_n \; = \; L_{h_1} z_n \; u_1 \; + \; u_2 \tag{12}$$

The original chained system has thus been converted to the following skew-symmetric chained system:

$$\begin{array}{rcl} \dot{z}_1 & = & u_1 \\ \dot{z}_2 & = & u_1 z_3 \\ \dot{z}_3 & = & -k_1 u_1 z_2 \; + \; u_1 z_4 \\ & \vdots & \\ \dot{z}_{j+3} & = & -k_{j+1} u_1 z_{j+2} \; + \; u_1 z_{j+4} \qquad (0 \leq j \leq n-4) \\ & \vdots & \\ \dot{z}_n & = & -k_{n-2} u_1 z_{n-1} + w_2 \end{array} \tag{13}$$

with:

$$w_2 \; = \; (k_{n-2} z_{n-1} + L_{h_1} z_n) u_1 \; + \; u_2 \tag{14}$$

The skew-symmetric nature of this form appears clearly when writing the system as follows:

$$
\dot{z}_1 = u_1
$$

$$
diag\{1, \frac{1}{k_1}, \ldots, \frac{1}{\prod_{j=1}^{j=n-2} k_j}\} \dot{Z}_2 = \begin{bmatrix} 0 & u_1 & 0 & & & \\ -u_1 & 0 & \frac{u_1}{k_1} & & & \\ 0 & -\frac{u_1}{k_1} & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \frac{u_1}{\prod_{j=1}^{j=n-3} k_j} \\ & & & & -\frac{u_1}{\prod_{j=1}^{j=n-3} k_j} & 0 \end{bmatrix} Z_2 + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \frac{1}{\prod_{j=1}^{j=n-2} k_j} \end{bmatrix} w_2 \tag{15}
$$

Skew-symmetric forms have been used, either explicitly or implicitly, by the author in his previous papers on mobile robot control. The interest of this form is that it naturally lends itself to Lyapunov control design and analysis, as illustrated by the following Proposition:

**Proposition 2.1**
*Assume that $|u_1(t)|$ and $|\dot{u}_1(t)|$ are bounded, and consider the control:*

$$
w_2 = -k_{w_2}(u_1) \, z_n \tag{16}
$$

*where $k_{w_2}(.)$ is a continuous application strictly positive on $R - \{0\}$.*
*If this control is applied to the system 13, then the positive function:*

$$
V(Z_2) = 1/2 \, ( \, z_2^2 + (1/k_1)z_3^2 + (1/k_1k_2)z_4^2 + \cdots + (1/\prod_{j=1}^{j=n-2} k_j)z_n^2 \, ) \tag{17}
$$

*is non-increasing along the closed-loop system's solutions, and asymptotically converges to some limit value $V_{lim}$ (which* a priori *depends on the initial conditions).*

*Moreover $u_1(t)V(Z_2(t))$ asymptotically tends to zero.*

*Therefore, if $u_1(t)$ does not asymptotically tend to zero, then $V_{lim} = 0$ and the origin $Z_2 = 0$ is globally asymptotically stable.*

**Proof of Proposition 2.1**:

Taking the time derivative of $V$ and using the system's $(n-1)$ last equations, one obtains:

$$\dot{V} \;=\; (1/\prod_{j=1}^{j=n-2} k_j)\; z_n w_2 \tag{18}$$

Thus, if the control 16 is used:

$$\dot{V} \;=\; -(k_{w_2}(u_1)/\prod_{j=1}^{j=n-2} k_j)\; z_n^2 \quad (\leq 0) \tag{19}$$

The considered Lyapunov-like function is thus non-increasing.

This in turn implies that $||Z_2(t)||$ is bounded, uniformly with respect to the initial conditions. Existence and uniqueness of the system's solutions over $R^+$ also follows.

Now, since $V$ is non increasing, $V(t)$ converges to some limit value $V_{lim}$ $(\geq 0)$. Since $k_{w_2}(.)$ is continuous, and since $|u_1(t)|$ and $|\dot{u}_1(t)|$ are bounded, $k_{w_2}(u_1(t))$ is uniformly continuous. Hence, the right-hand side member of equality 19 is uniformly continuous along any system's solution, and, by application of Barbalat's lemma, $\dot{V}(t)$ tends to zero. Therefore, $k_{w_2}(u_1(t))z_n(t)$ tends to zero. This in turn implies, using the properties of the function $k_{w_2}(.)$ and the boundedness of $|u_1(t)|$ and $|z_n(t)|$, that $u_1(t)z_n(t)$ tends to zero.

From now on, the time index will often be omitted to simplify the notations. Taking the time derivative of $u_1^2 z_n$, and using the convergence of $u_1 z_n$ to zero, gives:

$$\frac{d}{dt}(u_1^2 z_n) \;=\; -k_{n-2}u_1^3 z_{n-1} + o(t) \qquad with \quad \lim_{t\to+\infty} o(t) = 0 \tag{20}$$

$u_1^3 z_{n-1}$ is uniformly continuous along a system's solution since its time derivative is bounded. Therefore, in view of 20, and since $u_1^2 z_n$ tends to zero, $\frac{d}{dt}(u_1^2 z_n)$ also tends to zero (by application of a slightly generalized version of Barbalat's lemma). Hence, $u_1^3 z_{n-1}$, and thus $u_1 z_{n-1}$, tend to zero.

Taking the time derivative of $u_1^2 z_j$ and repeating the above procedure iteratively, one obtains that $u_1 z_j$ tends to zero for $2 \leq j \leq n$. In view of the system's equations, we note that this in turn implies the convergence of $\dot{Z}_2$ to zero.

Summing up the squared values of $u_1(t)z_j(t)$, for $2 \leq j \leq n$, it appears that $u_1(t)^2 V(t)$ tends to zero. And so does $u_1(t)^2 V_{lim}$ (from the already established convergence of $V(t)$ to $V_{lim}$).

$\diamond$

**Remarks**:

- One can easily verify that, with the particular choice $k_{w_2}(u_1) = k'_{w_2}|u_1|$ ($k'_{w_2}$ being a positive real number), the control $u_2$ given by 14 and 16 coincides with the "linearizing" control 7 associated with the linear invariant system 4. More precisely, there is a one-to-one correspondance between the parameters of the two controls. One can thus apply classical linear control methods to determine these parameters and optimize the control performance near the point $Z_2 = 0$, as illustrated in [27].

- Non-convergence of $u_1(t)$ to zero, under the assumption that $|\dot{u}_1(t)|$ is bounded, implies that $\int_0^t |u_1(\tau)|d\tau$ tends to infinity with $t$. Divergence of this integral is in fact necessary to the asymptotical convergence of $||Z_2(t)||$ to zero, when using the control 16 with $k_{w_2}(u_1) = k'_{w_2}|u_1|$. This appears clearly when interpreting this control as a stabilizing linear control for the linear invariant system 4 obtained by replacing the time variable by the aforementionned integral. However, this integral may still diverge when $u_1(t)$ tends to zero slowly "enough" (like $t^{-\frac{1}{2}}$, for example). This indicates that $||Z_2(t)||$ may still converge to zero when $u_1(t)$ does.

Proposition 2.1 is not only of interest to solve the path following problem for mobile robots, as this will be illustrated further on, it also yields a simple way of determining smooth time-varying feedbacks which globally asymptotically stabilize the origin $Z = 0$ (or $X = 0$) of the complete system. In this case, $u_1$ is used as a control the role of which is to complement the action of the control $u_2$ (or $w_2$) in order to also obtain asymptotical convergence of $z_1$ (or $x_1$) to zero. Since it is known that chained systems like 1, with $n \geq 3$, cannot be asymptotically stabilized by using smooth pure state feedbacks (by application of a Brockett's theorem [5]), smooth feedback stabilization can only be achieved by using a time-varying control law.

**Proposition 2.2**

*Consider the same control as in Proposition 2.1:*

$$w_2 = -k_{w_2}(u_1)\, z_n \tag{21}$$

*complemented with the following time-varying control:*

$$u_1 = -k_{u_1}\, z_1 + h(Z_2, t) \tag{22}$$

*where:*

- $k_{u_1}$ *is a positive real number;*

- $h(Z_2, t)$ *is a function of class* $C^{p+1}$ $(p \geq 1)$*, uniformly bounded with respect to* $t$*, with all successive partial derivatives also uniformly bounded with respect to* $t$*, and such that:*

**$C_1$**: $h(0,t) \ = \ 0 \quad ,\forall t$

**$C_2$**: *There is a time-diverging sequence $\{t_i\}_{i\in N}$, and a positive continuous function $\alpha(.)$ such that:*

$$||Z_2|| \ \geq \ l \ > 0 \implies \sum_{j=1}^{j=p}\left(\frac{\partial^j k}{\partial t^j}(Z_2,t_i)\right)^2 \ \geq \alpha(l) \ > \ 0 \qquad ,\forall i$$

*The controls 21 and 22 globally asymptotically stabilize the origin $Z = 0$.*

**Proof of Proposition 2.2**:

It has already be shown that the positive function $V(Z_2)$ used in Proposition 2.1 is non-increasing along the closed-loop system's solutions, implying that $||Z_2||(t)$ is bounded uniformly with respect to initial conditions.

The first equation of the controlled system is:

$$\dot{z}_1 \ = \ -k_{u_1}\, z_1 \ + \ h(Z_2,t) \tag{23}$$

This is the equation of a stable linear system subjected to the bounded additive perturbation $h(Z_2(t),t)$. Therefore, $|z_1(t)|$ is also bounded uniformly with respect to the initial conditions.

Existence and unicity of the solutions over $R^+$ is thus ensured.

From the expression of $u_1$, it is then found that $u_1(t)$ (taken as a function of time along a system's solution) is bounded. And so is its first derivative (by using the regularity properties imposed upon $h(Z_2,t)$).

Proposition 2.1 thus applies. In particular, $V(Z_2(t))$ tends to some limit value $V_{lim}$ $(\geq 0)$, $||\dot{Z}_2(t)||$ tends to zero, and $Z_2(t)$ tends to zero if $u_1(t)$ does not.

We now proceed by contradiction.

Assume that $u_1(t)$ does not tend to zero. Then, $||Z_2(t)||$ tends to zero. By uniform continuity, and since $h(0,t) = 0$ (condition $C_1$), $h(Z_2(t),t)$ also tends to zero. Equation 23 then becomes the equation of a stable linear system subjected to an additive perturbation which asymptotically vanishes. As a consequence, $z_1(t)$ tends to zero. From the expression of $u_1$, this in turn implies that $u_1(t)$ tends to zero, yielding a contradiction.

Therefore, $u_1(t)$ must asymptotically tend to zero.

Differentiating the expression of $u_1$ with respect to time, and using the convergence of $u_1(t)$ and $||\dot{Z}_2(t)||$ to zero, we get:

$$\dot{u}_1(t) \ = \ \frac{\partial h}{\partial t}(Z_2(t),t) + o(t) \qquad with \quad \lim_{t\to+\infty} o(t) = 0 \tag{24}$$

Since $\frac{\partial h}{\partial t}(Z_2(t),t)$ is uniformly continuous (its time derivative is bounded), $\dot{u}_1(t)$, and thus $\frac{\partial h}{\partial t}(Z_2(t),t)$, tend to zero (Barbalat's lemma).

By using similar arguments, one obtains that the time-derivative of $\frac{\partial h}{\partial t}(Z_2(t),t)$ and $\frac{\partial^2 h}{\partial t^2}(Z_2(t),t)$ tend to zero.

By repeating the same procedure as many times as necessary, we show that $\frac{\partial^j h}{\partial t^j}(Z_2(t),t)$ tends to zero, $(1 \leq j \leq p)$. Therefore:

$$\lim_{t \to \infty} \sum_{j=1}^{j=p} (\frac{\partial^j k}{\partial t^j}(Z_2(t),t))^2 \; = \; 0 \tag{25}$$

Assume now that $V_{lim}$ is different from zero. This implies that $||Z_2(t)||$ remains larger that some positive real number $l$ (which can be calculated from $V_{lim}$). The previous convergence result is then not compatible with the condition $C_2$ imposed on the function $h(Z_2,t)$.

Hence, $V_{lim}$ is equal to zero and $Z_2(t)$ asymptotically converges to zero. Then, by uniform continuity and using the condition $C_1$, $h(Z_2(t),t)$ tends to zero.

In view of the expression of $u_1$, asymptotical convergence of $z_1(t)$ to zero readily follows.

$\diamond$

**Remark:** Dependence of the function $h(Z_2,t)$ upon the last state variable $z_n$ is not required when the function $k_{w_2}(.)$ is strictly positive on $R$. The reason is that $z_n(t)$ unconditionally converges to zero in this case, due to the convergence of $V(Z_2(t))$ to zero (cf. proof of Proposition 2.1).

It can be noted that only the control input $u_1$ depends on time explicitly via the function $h(Z_2,t)$. We will refer to this function as the **heat-function**, in order to establish a parallel with well known probabilistic global minimization methods, and underline the primary role of this term in the control, i.e. forcing "motion" as long as the system has not reached the desired equilibrium point, thus preventing the system's state from converging to other equilibrium points.

According to Proposition 2.1, when one is only interested in the regulation of $Z_2$ (as in case of mobile robot path following), any sufficiently regular input $u_1(t)$, which does not asymptotically tend to zero, can be used. This leaves the user with some freedom concerning the choice of this input. For instance, uniform exponential convergence of $||Z_2(t)||$ to zero is obtained when $|u_1(t)|$ remains larger than some positive number. Other sufficient conditions for exponential convergence of $||Z_2(t)||$ to zero, which do not require $u_1(t)$ to have always the same sign, may also be derived. For example, if $|\dot{u}_1(t)|$ is bounded, it is sufficient to have $|u_1(t)|$ periodically larger than some positive number.

If the application further requires the regulation of $z_1$ (stabilization of a mobile robot about a fixed desired configuration, for example), then Proposition

2.2 suggests implementing a time-varying feedback $u_1$. In both cases, the same control law $u_2$ (or $w_2$) based on $u_1$-*time-scaling* can be used.

The conditions imposed by Theorem 2.2 upon the *heat-function* are not severe and can easily be met. For example, the following three functions:

$$h(Z_2,t) \;=\; ||Z_2||^2 sin(t) \tag{26}$$

$$h(Z_2,t) \;=\; \sum_{j=0}^{j=n-2} a_j sin(\beta_j t) z_{2+j} \tag{27}$$

$$h(Z_2,t) \;=\; \sum_{j=0}^{j=n-2} a_j \frac{exp(b_j z_{2+j})-1}{exp(b_j z_{2+j})+1} sin(\beta_j t) \tag{28}$$

(with $a_j \neq 0$, $b_j \neq 0$, $\beta_j \neq 0$, and $\beta_i \neq \beta_j$ when $i \neq j$) satisfy these conditions. For the first function, this is obvious. For the second function, the proof can be found in [25]. The same proof basically applies to the third function which presents the additional feature of being uniformly bounded with respect to all its arguments. It can be noted that it is not necessary to use sinusoids at integrally related frequencies, as opposed to the solution proposed in [31]. In fact, the theorem indicates that it is not even necessary to use time-periodic functions, as it is assumed in most time-varying feedback stabilization studies.

For practical purposes, the choice of the *heat-function* is important because the overall control performance (asymptotical convergence rate, time needed to enter a small ball centered on zero, sensitivity with respect to perturbations,...) critically depends upon this choice. This has been checked by the author in simulation. An analysis, given further, also explains why the functions 27 and 28 are better than 26 with respect to the induced asymptotical convergence rate. For the last two functions, the parameters $a_j$ and $b_j$, which characterize the "slope" of $h(Z_2,t)$ near the origin $Z_2 = 0$, have been found to have much influence on the transient time needed for the system's solutions to get close to zero. Basically, the larger these parameters are, the shorter the transient time is.

## 2.3 Continuous time-varying feedbacks and exponential convergence in the 3-dimensional case

In the previous section, the time-varying feedback laws proposed for point-stabilization are at least of class $C^1$ when the function $k_{w_2}(.)$ involved in the expression of the control $w_2$ is itself differentiable on $R$.

We show in this section that the above Lyapunov-based approach still applies, at least in the 3-dimensional case, to the design of stabilizing continuous feedbacks which are not differentiable at zero. A subclass of **exponentially**

**convergent** time-varying feedbacks is also derived, using ideas presented in a recent convergence rate study by M'Closkey and Murray in [14]

Let us thus consider the three dimensional chained-form system:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_1 x_3 \\ \dot{x}_3 &= u_2 \end{aligned} \tag{29}$$

and the usual "sign" function:

$$sign(x) \;=\; \begin{cases} 1 & if \;\; x \geq 0 \\ -1 & if \;\; x < 0 \end{cases} \tag{30}$$

We have the following result, which is quite alike Proposition 2.1 in the three-dimensional case except that a different Lyapunov-like function, associated with a different control law, is used:

**Proposition 2.3**
*If $u_1(t)$ is uniformly continuous and $|u_1(t)|$ is bounded, and if the feedback control:*

$$u_2 \;=\; -k_3 u_1 sign(x_2)(x_2^2)^{\frac{a-2}{2a}} - k_4 x_3 \qquad with\ k_3 > 0,\ k_4 > 0, (a \in N,\ a \geq 2) \tag{31}$$

*is applied to the system 29, then the positive function:*

$$V(x_2, x_3) \;=\; \frac{1}{2}(({x_2}^2)^{\frac{a-1}{a}} + \frac{a-1}{k_3 a}{x_3}^2) \tag{32}$$

*is non-increasing along any system's solution, and asymptotically converges to some limit value $V_{lim}$. Moreover, $u_1(t)V(t)$ asymptotically tends to zero.*

*Therefore, if $u_1(t)$ does not asymptotically tend to zero, then $V_{lim} = 0$, which in turn implies that $x_2(t)$ and $x_3(t)$ asymptotically tend to zero.*

**Proof of theorem 2.3**:
Note that the function $V(x_2, x_3)$ is everywhere differentiable except at $x_2 = 0$ when $a = 2$. The control law $u_2(x_2, x_3, t)$ is also everywhere continuous, except on the manifold $x_2 = 0$ when $a = 2$. Proving the existence and uniqueness of the system's solutions is thus slightly more difficult in the particular case where $a = 2$. This technical difficulty will left be aside here.

Differentiating $V(x_2, x_3)$ with respect to time along a system's solution gives:

$$\dot{V} \;=\; \frac{a-1}{a} x_3 [u_1 sign(x_2)(x_2^2)^{\frac{a-2}{2a}} + \frac{u_2}{k_3}] \tag{33}$$

and, replacing $u_2$ by its expression 31:

$$\dot{V} = -\frac{a-1}{a}\frac{k_4}{k_3}{x_3}^2 \tag{34}$$

The time derivative of $V(t)$ is thus always defined and non-positive, which proves that $|x_2(t)|$ and $|x_3(t)|$ are uniformly bounded with respect to initial conditions. Also, $V(t)$ is bounded and $V(t)$ converges to some limit value denoted as $V_{lim}$. Moreover, since ${x_3}^2(t)$ is uniformly continuous (its time derivative is bounded), $\dot{V}(t)$ asymptotically converges to zero (Barbalat's lemma). Hence, $x_3(t)$ tends to zero.

The product $x_2(t)x_3(t)$ also tends to zero. Taking its time derivative gives:

$$\frac{d}{dt}(x_2x_3) = u_1x_3^2 - k_4x_2x_3 - k_3u_1(x_2^2)^{\frac{a-1}{a}} \tag{35}$$

It is uniformly continuous along a system' solution, since all terms involved in the right-hand side member of the previous equality are bounded and uniformly continuous. It thus tends to zero, by Barbalat's lemma. Using the convergence of $x_3(t)$ to zero, this in turn implies that $u_1(t)x_2(t)$ tends to zero. Combining this result with the fact that $u_1(t)x_3(t)$ tends to zero, one obtains that $u_1(t)V(t)$ tends to zero.

$\diamond$

As in the previous section, it is not difficult to derive from there a *time-varying* feedback control $u_1$ which, combined with the above control $u_2$, globally asymptotically stabilizes the origin $(x_1 = 0, x_2 = 0, x_3 = 0)$. A subset of such controls, having the additional property of making the system's solutions converge **exponentially** to zero, is proposed next.

**Proposition 2.4**

*Consider the control $u_2$ used in Proposition 2.3 complemented with the following time-varying control:*

$$u_1 = -k_1\ x_1 + h(x_2, t) \qquad ; \ k_1 > 0 \tag{36}$$

*where the heat-function $h(x_2, t)$ is a continuous function defined on $R \times R^+$, twice differentiable everywhere except may be on the manifold $x_2 = 0$, uniformly bounded with respect to $t$, and with partial derivatives also uniformly bounded with respect to $t$ when $x_2 \neq 0$.*

*Two additional conditions are imposed upon the choice of this function:*

$\mathbf{C_1}$*:* $h(0, t) = 0 \ , \forall t$

$\mathbf{C_2}$*:* $\forall x_2 \neq 0$, $\frac{\partial h}{\partial t}(x_2, t)$ *does not asymptotically tend to zero when $t$ tends to infinity*

*Then:*

**i)** *The origin $X = 0$ of the controlled system is globally asymptotically stable.*

**ii)** *With the particular choice:*

$$h(x_2, t) \;=\; k_2 \; sign(x_2) \; (x_2^2)^{\frac{1}{2a}} \; sin(\omega t) \qquad ; \; \omega \neq 0 \tag{37}$$

*the origin $X = 0$ is* $\rho$-exponentially stable *with respect to the* homogeneous norm $\rho(X) = (x_1^{2a(a-1)} + x_2^{2(a-1)} + x_3^{2a})^{\frac{1}{2a(a-1)}}$ *in the sense that:*

$$\exists M > 0, \; \exists \alpha > 0 \;\; : \;\; \rho(X(t)) \;\leq\; M \; \rho(X(0)) \; exp(-\alpha t), \quad \textit{for all } t \geq 0 \tag{38}$$

**Proof of Proposition 2.4**:

Since the same control $u_2$ as in Proposition 2.3 is used, the boundedness of $|x_1(t)|$ and $|\dot{x}_1(t)|$ follows from the previously established boundedness of $V(x_2(t), x_3(t))$ and from the first system's equation which can be interpreted as the equation of a stable linear system perturbed by a bounded term.

As a consequence, $|x_1(t)|$ is bounded along any system's solution, implying in turn that $|u_1(t)|$ and $|\dot{x}_2(t)|$ are bounded.

One can show from there, as done in the proof of Proposition 2.3, that $\dot{V}(t)$ and $x_3(t)$ tend to zero, while $V(t)$ tends to some limit value denoted as $V_{lim}$. Therefore, $(x_2(t)^2)^{\frac{a-1}{a}}$ tends to $2V_{lim}$, implying that $x_2(t)$ tends itself to some limit value $x_{2,lim}$.

Let us assume that $x_{2,lim}$ is not equal to zero. There exists a time $t_0$ such that $|x_2(t)| > \epsilon > 0$ if $t > t_0$. From the expression of $u_1$ and the conditions imposed upon $h(x_2, t)$, it then appears that $u_1(t)$ is in this case uniformly continuous when $t > t_0$. The results of Proposition 2.3 then apply. In particular, $u_1(t)$ must tend to zero (since otherwise $x_2(t)$ would tend to zero, yielding a contradiction). From the expression of $u_1$, it then comes that $z(t) = -k_1 x_1(t) + h(x_{2,lim}, t)$ tends to zero. The time-derivative of $z(t)$:

$$\frac{d}{dt} z(t) \;=\; -k_1 u_1(t) + \frac{\partial h}{\partial t}(x_{2,lim}, t)$$

is uniformly continuous when $t > t_0$, and thus tends to zero (Barbalat's lemma). Hence, $-k_1 u_1(t) + \frac{\partial h}{\partial t}(x_{2,lim}, t)$ tends to zero. From the condition $C_2$ imposed on the heat function, $\frac{\partial h}{\partial t}(x_{2,lim}, t)$ does not tend to zero in this case. Therefore $u_1(t)$ does not either tend to zero, yielding a contradiction.

$x_{2,lim}$ is thus necessarily equal to zero. Asymptotical convergence of $x_1(t)$ to zero then follows immediately from the first system's equation and from the condition $C_1$.

This concludes the proof of result i).

The second result is inspired from a recent work by M'Closkey and Murray [14] who have had the idea of utilizing Kawski's results on homogeneous systems [12] to study the asymptotical convergence rate of so-called *power form* systems subjected to stabilizing time-varying feedbacks. In the three-dimensional case, the power form and the chain form are in fact the same.

Let us only recall here, by using the same notations as in [14], that a system $\dot{X} = f(X)$ of dimension $n$ is said to be *homogeneous of order zero* with respect to the *dilation* operator $\delta_\lambda X = (\lambda^{k_1}x_1, \cdots, \lambda^{k_n}x_n)^T$, where the $k_i$'s are positive integers, if $\delta_\lambda \dot{X} = f(\delta_\lambda X)$, $\forall \lambda > 0$. For example, linear systems are homogeneous of order zero with respect to the dilation operator $\delta_\lambda X = (\lambda x_1, \cdots, \lambda x_n)^T$.

A *homogeneous norm* associated with this type of system is a positive operator in the form $\rho(X) = (\sum_{i=1}^{i=n} |x_i|^{p/k_i})^{1/p}$ which satisfies the property $\rho(\delta_\lambda X) = \lambda \rho(X)$. In the linear case, such a norm is the usual Euclidean norm. Note that, in the general case, this norm is not equivalent to the usual $p$-norms.

It is then possible to define the notion of *$\rho$-exponential stability* as follows: the equilibrium point $X = 0$ of a differential equation $\dot{X} = f(X)$ is called $\rho$-exponentially stable if there is a neighbourhood $\mathcal{U}$ of $X = 0$ and constants $M$ and $\alpha$ such that:

$$\rho(X(t)) \leq M\ \rho(X(0))\ exp(-\alpha t), \quad for\ all\ t \geq 0, \quad X(0) \in \mathcal{U}$$

The main result of interest here, pointed out by Kawski ([12], Lemma 2.1, pp. 502), is that, for such a system with $f(X)$ being continuous, the origin $X = 0$ is **asymptotically stable if and only if it is $\rho$-exponentially stable**. In other words, asymptotical stability implies $\rho$-exponential stability, just as in the linear case. We will assume here, as in the work of M'Closkey and Murray, that this result extends to the case of time-periodic systems. This is probably true, but we must concede that a rigorous verification of this point is still needed.

Coming back to the proof of the Proposition's point ii), it is simple to verify that the controlled system, with the considered heat function, is homogeneous of order zero with respect to the dilation $\delta_\lambda X = (\lambda x_1, \lambda^a x_2, \lambda^{a-1}x_3)$, and that an associated homogeneous $\rho$-norm is the one considered in the Proposition. Since asymptotical stability has already been proven, the second point of Proposition 2.4 just follows from the aforementionned Kawski's result.
$\diamond$

The possibility of extending this type of result to systems of higher dimension will be the subject of future studies.

Data from a simulation of a continuous $\rho$-exponentially stabilizing feedback, in the case of a unicycle-type vehicle, and with the parameter $a$ set equal to three, are presented in the **Figures 1.a, 1.b**, and **1.c**. The simulated control law takes into account the modification proposed in Section 3.4.

This section provides us with elements which will be used in the forthcoming discussion concerning the asymptotical rate of convergence associated with various stabilizing feedback control schemes proposed in the literature for nonholonomic wheeled mobile robots.

## 2.4 Asymptotical rate of convergence

From Section 2.2, we already know that, when using the control law 21, 22 with $k_{w_2}(u_1) = k'_{w_2}|u_1|$ ($k'_{w_2} > 0$), the convergence of $||Z(t)||$ to zero **cannot be exponential**. Indeed, $u_1(t)$ would otherwise converge to zero exponentially and the integral $\int_0^t |u_1(\tau)|d\tau$ would not diverge. This would be in contradiction with the fact (pointed out earlier) that divergence of this integral is necessary to the asymptotical convergence of $||Z_2(t)||$ to zero.

From the simulation of a smooth time-varying feedback control applied to a unicycle, it has also been observed in [26] that the norm of the state vector did not converge to zero faster than $t^{-\frac{1}{2}}$ for most initial configurations. This is much slower than the exponential rate of convergence that can be obtained in the case of nonlinear systems the linear tangent approximation of which is controllable. It is claimed in [11] that it is not possible to achieve *exponential stability* for nonholonomic systems by using smooth (differentiable everywhere) time-periodic feedbacks. In mathematical terms, this means that the sytem's trajectories cannot satisfy the following inequality:

$$||X(t)|| \;\leq\; K\;||X(0)||\;exp(-\lambda t) \qquad \forall X(0) \textit{ in some open ball centered on zero} \tag{39}$$

for some positive real numbers $K$ and $\lambda$.

The practical significance of this relation, when it is satisfied, is twofolds: i) small initial errors cannot produce large transient deviations since $||X(t)|| \;<\; K\;||X(0)||$, and ii) all solutions end up converging to zero exponentially.

It is worth noting that these two properties, regrouped under the strong concept of exponential stability, do not necessarily hold together.

For example, the piecewise-continuous time-invariant feedback law proposed by Canudas and Sordalen in [6], for posture stabilisation of a unicycle, only yields the following result:

$$||X(t)|| \;\leq\; (K_1 + K_2||X(0)||)\;exp(-\lambda t) \tag{40}$$

Each solution converges to zero exponentially, but the slightest initial error, or perturbation, may produce transient deviations the size of which is larger than some constant.

In his doctoral dissertation [29], Sordalen proposes another interesting control which may be seen as a time-varying mixt of open-loop and feedback strategies, continuous with respect to time, but non-smooth in the state vector. He shows that this control, which applies to any chained-form system, yields the following property:

$$||X(t)|| \ \leq \ g(||X(0)||) \ \ exp(-\lambda t) \tag{41}$$

where $g(.)$ is a *class $\mathcal{K}$-function* (i.e. strictly increasing and such that $g(0) = 0$) which is not lipschitz around zero. Precisely, the derivative of $g(x)$ tends to infinity when $x$ tends to zero.

This property has been called *$\mathcal{K}$-exponential stability*. It is weaker than the usual exponential stability notion in the sense that small initial errors (or perturbations) can produce transient deviations of much larger amplitude. Nethertheless, it is better than 40 in the sense that the deviations are not lowerbounded by some positive constant. As in the previous case, all solutions tend to zero exponentially in the absence of pertubations acting on the system.

In the previous section, a set of continuous time-varying feedbacks, not differentiable at the origin, have been shown to share the same type of stability property, in the three-dimensional case.

Concerning the case of smooth time-varying feedbacks, such as the ones derived in the Section 2.2, it is simple to verify that we have:

$$||X(t)|| \ \leq \ K \ ||X(0)|| \tag{42}$$

for some positive constant $K$.

Indeed, from Proposition 2.1, and using the letter $X$ instead of $Z$:

$$||X_2(t)|| \ \leq \ K_1 V(X_2(t)) \ \leq \ K_1 V(X_2(0)) \ \leq \ K_2 ||X_2(0)|| \tag{43}$$

Using the properties of the function $h(X_2, t)$ involved in the control expression:

$$|h(X_2(t), t)| \ \leq \ K_3 ||X_2(t)|| \ \leq \ K_2 K_3 ||X_2(0)|| \tag{44}$$

and from equation 23 (with $x_1$ instead of $z_1$):

$$|x_1| \ \leq \ sup(|x_1(0)|, \frac{K_2 K_3}{k_{u_1}} ||X_2(0)||) \tag{45}$$

Relation 42 is thus satisfied with $K = K_2 + sup(1, \frac{K_2 K_3}{k_{u_1}})$.

For example, with all control gains $k_i$ and $k_{u_1}$ taken equal to one, and a function $h(X_2, t)$ such that $K_3 = 1$, then $K_2 = 1$ and $K = 2$.

This establishes that smooth time-varying feedbacks are, in some sense, less sensitive to initial errors than the aforementionned non-smooth feedbacks. The "price" paid for this type of robustness is that the system's solutions do not converge to zero as fast as exponentially. In fact, in view of the above discussion, one can only expect to have:

$$||X(t)|| \leq K \, ||X(0)||f(t) \qquad ; \; f(0) = 1 \; , \; \lim_{t \to +\infty} f(t) = 0 \tag{46}$$

where $f(t)$ is a decreasing function which does not tend to zero as fast as exponentially. For example: $f(t) = (1+t)^{-\frac{1}{2}}$, in the case of the control considered in [26].

The purpose of the above discussion was to summarize our actual knowledge concerning the stability properties of controlled nonholonomic systems in the case of point-stabilization, and to point out the difficulty to objectively compare known, smooth and non-smooth, feedback solutions. So far, exponential stability, in the usual sense, of nonholonomic vehicles has not been obtained and is likely out of reach. Non-smooth (time-invariant or time-varying) feedback controls yielding exponential convergence of the solutions to zero, in the absence of perturbations and modelling errors, appear to be somewhat sensitive to initial conditions and perturbations in the vicinity of zero. Smooth time-varying feedbacks are less sensitive, but are also less efficient in terms of asymptotical rate of convergence. This still does not mean that they cannot steer the system's solutions to an arbitrarily small neighbourhood of zero as fast as other controls, as pointed out in [31] (for example) and illustrated by simulation results in [27]. It should also be noted, that perturbations acting on nonholonomic systems are not of equal importance depending on the state component which is primarily affected: a deviation in a direction compatible with the vehicle's mobility is clearly not as severe as a deviation which violates one of the system's kinematic constraints (lateral motion of a car, for example). Further clarification of these issues is thus needed.

The remaining of this section is devoted to a short (and not yet rigorous) analysis attempting to account for the asymptotical rate of convergence like $t^{-\frac{1}{2}}$ observed in simulations when applying a smooth time-varying feedback to a nonholonomic unicycle-type vehicle.

Take the three-dimensional skew-symmetric chained-form control system into which the vehicle's kinematic equations can be transformed:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_1 x_3 \\ \dot{x}_3 &= -k_1 u_1 x_2 + w_2 \end{aligned} \tag{47}$$

and a smooth time-varying feedback law belonging to the set of controls considered in Proposition 2.2:

$$\begin{aligned} u_1 &= -k_{u_1} x_1 + h(x_2, t) \\ w_2 &= -k_{w_2} x_3 \end{aligned} \tag{48}$$

Assuming that the control gains $k_{u_1}$ and $k_{w_2}$ are large (which should not modify the asymptotical rate of convergence of the solutions), the first and third system's equations suggest the following approximations:

$$x_1 \approx \frac{1}{k_{u_1}} h(x_2, t) \tag{49}$$

$$x_3 \approx -\frac{k_1}{k_{w_2}} u_1 x_2 \tag{50}$$

The second approximation indicates that $|x_3|$ asymptotically decreases faster than $|x_2|$, since $u_1$ also tends to zero.

Using this relation in the second system's equation:

$$\dot{x}_2 \approx -\frac{k_1}{k_{w_2}} {u_1}^2 x_2 \tag{51}$$

which shows how the rate of convergence of $x_2$ to zero depends upon the size of $|u_1|$. If $|u_1|$ remained periodically larger that some positive constant, then, according to 51, $x_2$ would converge to zero exponentially.

Taking the time derivatives of both members of 49:

$$u_1 \approx \frac{1}{k_{u_1}} [\frac{\partial h}{\partial x_2} u_1 x_3 + \frac{\partial h}{\partial t}] \tag{52}$$

and since $\frac{1}{k_{u_1}} \frac{\partial h}{\partial x_2} x_3$ becomes small in front of one:

$$u_1 \approx \frac{1}{k_{u_1}} \frac{\partial h}{\partial t}(x_2, t) \tag{53}$$

Using the above expression of $u_1$ in 51:

$$\dot{x}_2 \approx -\frac{k_1}{k_{w_2} k_{u_1}^2} (\frac{\partial h}{\partial t}(x_2, t))^2 x_2 \tag{54}$$

This relation is of interest because it shows how that the convergence rate of $x_2$ to zero is connected to the choice of the heat-function $h(x_2, t)$.

Take, for example, a periodic function such as:

$$h(x_2, t) = {x_2}^p sin(t) \tag{55}$$

Normally, $p$ should not be smaller than one in order to ensure differentiability at the origin (as required in Theorem 2.2).

Then, equation 54 becomes:

$$\dot{x}_2 \approx -\frac{k_1}{k_{w_2} k_{u_1}^2} {x_2}^{2p+1} cos(t)^2 \tag{56}$$

Replacing $cos(t)^2$ by its average value 0.5, the above equation simplifies as:

$$\dot{x}_2 \approx -\frac{k_1}{2k_{w_2}k_{u_1}^2} \, {x_2}^{2p+1} \tag{57}$$

Solutions to this equation are of the form (omitting a constant multiplicative factor):

$$x_2(t) \sim t^{-\frac{1}{2p}} \tag{58}$$

Combining this result with 49 and 55:

$$x_1(t) \sim t^{-\frac{1}{2}} \tag{59}$$

and, from 50, 53, and 58:

$$x_3(t) \sim t^{-\frac{p+1}{2p}} \tag{60}$$

Relations 58-60 are the main outputs of this sketch-analysis (which may be worked out more rigorously either by applying Centre Manifold techniques [7], or by invoking two-time scale techniques as done in [14]). They give an estimation of each state variable worst asymptotical rate of convergence as a function of the parameter $p$ which characterizes the way the heat-function $h(x_2, t)$ converges to zero with $x_2$.

In all cases, $x_1(t)$ tends to zero like $t^{-\frac{1}{2}}$, so that the worst asymptotical rate of convergence of $||X(t)||$ cannot be faster than $t^{-\frac{1}{2}}$. This rate is reached when $p$ is equal to (or smaller than) one, as in the simulation presented in [26].

If $p$ is larger than one, then $x_2(t)$ does not converge to zero as fast as $x_1(t)$, and convergence of $||X(t)||$ is slower (like $t^{-\frac{1}{2p}}$).

If $p$ is smaller than one, then the analysis suggests that $x_2(t)$ and $x_3(t)$ can converge to zero significantly faster than $t^{-\frac{1}{2}}$, although still not as fast as exponentially. Simulations performed by the author tend to confirm this point. However, the function $h(x_2, t)$ is no longer differentiable on the manifold $x_2 = 0$, and the time-varying feedback control law is only continuous in this case. As a consequence, the property 42 is also lost.

Another step is taken by choosing $p$ equal to zero ($\Rightarrow h(t) = sint$). Then, according to 51, $x_2(t)$ converges to zero exponentially, and so does $x_3(t)$. But, $x_1(t)$ does not converge to zero in this case, due to the fact that $h(x_2, t)$ does not converge to zero when $x_2$ does.

The critical role played by the function $h(x_2, t)$ clearly appears in the above analysis. Exponential convergence of $x_2$ and $x_3$ requires a function which does not tend to zero, while convergence of $x_1$ to zero is possible only when this function does tend to zero. Exponential convergence of all state variables to zero thus seems to be impossible with this type of control.

This in turn raises a rather fundamental question, seldom addressed in the control literature: **is the asymptotical rate of convergence a good measurement of the overall control performance?** Answering this question is not simple, knowing that regulation errors are physically unavoidable and that what often really matters in practice is to keep these errors as small as possible under realistic adverse experimental conditions. While connections between robustness issues and asymptotical rate of convergence of the controlled system have been much studied in the case of linear systems (or nonlinear systems that can be approximated by controllable linear systems), they are still not well understood in other cases.

To illustrate the difficulty with a concrete example, a control law similar to 48 has been simulated for a unicycle-type vehicle with the following *non-smooth* heat-function:

$$h(x_2, x_3, t) \; = \; \begin{cases} sin(t) & if \; x_2^2 + (1/k_1)x_3^2 \; \geq \; \epsilon^2 \\ 0 & if \; x_2^2 + (1/k_1)x_3^2 \; < \; \epsilon^2 \end{cases} \tag{61}$$

Note that this function does not satisfies the conditions imposed in Proposition 2.2. The corresponding feedback control is time-varying, but not even continuous.

The $(x_1(t), x_2(t))$ cartesian position of the vehicle is represented in **Fig. 2.a**.

The time-evolution of $x_1(t)$, $x_2(t)$, and the vehicle's orientation angle $\theta(t)$ ($\approx x_3(t)$) is shown in **Fig. 2.b**.

After 25 seconds, all variables "seem" to have converged to zero. In reality, it is possible to show that $x_1(t)$ and both control inputs converge to zero exponentially, while $(x_2(t)^2 + (1/k_1)x_3(t)^2)$ can only be shown to become smaller than $\epsilon^2$ ($= 10^{-6}$, in the simulation) after a finite time (see **Fig. 2.c**, and **2.d**).

This control thus does not asymptotically stabilizes the system about the desired equilibrium, so that the notion of asymptotical rate of convergence does not even apply here. Is this fact sufficient to assert that this is not a good control?

# 3 Application to the control of a car with -n-trailers

## 3.1 Modelling equations and notations

We consider a car with $-n-$ trailers as represented in **Fig. 3**. The system is assumed to move on flat ground. The wheels are allowed to roll and spin, but not slip.

**The vehicle counted first is the last trailer** and the following notations are used:

- The $ith$ vehicle is attached to the next vehicle at a point $P_i$ located on the rear wheels'axle of the next vehicle, and it is assumed that $P_{i-1}P_i$ is orthogonal to the wheels' axle of the $ith$ vehicle. The point $P_{n+1}$ is a point located on the vertical axis of the car's steering wheel, and $P_nP_{n+1}$ is assumed to be orthogonal to the car rear-wheels' axle.
- $l_i$ is the distance between $P_i$ and $P_{i+1}$
- $\alpha_i$ $(1 \leq i \leq n)$ is the angle between $P_{i-1}P_i$ and $P_iP_{i+1}$ which characterizes the orientation of the vehicle $(i+1)$ with respect to the previous vehicle
- $\alpha_0$ gives the orientation of the first vehicle with respect to a fixed frame. For instance, we may choose: $\alpha_0 = angle(\vec{i_0}, \vec{P_0P_1})$
- $\alpha_{n+1}$ is the angle of the car's driving front wheel with respect to the car's body.
- $v_i$ $(0 \leq i \leq n+1)$, is the intensity of the velocity of the point $P_i$. This is the translational velocity of the $(i+1)$ vehicle.
- $r$ is the radius of the car's front steering wheel, and $\omega$ the angular velocity of this wheel about its horizontal axis so that $v_{n+1} = r\omega$.

In what follows, in order to simplify the study and focus on the core of the problems arising when trying to control nonholonomic systems, only velocity control is considered, and $\omega$ and $\frac{d}{dt}\alpha_{n+1}$ are chosen as control variables.

Kinematic equations of this system have been derived by various authors. See [13], [20], [32], for example.

A first set of equations is simply obtained by using the classical identity:

$$\frac{d}{dt}\vec{P_{i+1}} \;=\; \frac{d}{dt}\vec{P_i} \;+\; \vec{\omega_i} \wedge \vec{P_iP_{i+1}} \qquad for\ 0 \leq i \leq n \tag{62}$$

where $\vec{\omega_i}$ is the angular velocity about the vertical axis of the $ith$ vehicle's body.

This yields the following equations:

$$\begin{array}{lcl} v_{n+1} & = & r\omega \\ v_i & = & v_{i+1}cos(\alpha_{i+1}) \qquad (0 \leq i \leq n) \\ \dot{\alpha}_0 & = & v_0\,\frac{tan(\alpha_1)}{l_1} \\ \dot{\alpha}_i & = & v_i(\frac{tan(\alpha_{i+1})}{l_{i+1}} - \frac{sin(\alpha_i)}{l_i}) \qquad (1 \leq i \leq n) \end{array} \tag{63}$$

We note that the set of variables $\alpha_i$ entirely characterizes the relative positioning of each vehicle with respect to the others. The remaining equations that are needed are thus the equations of motion of one of the vehicles relatively to the path we would like this vehicle to follow. To this purpose, we choose the first vehicle (i.e. the last trailer) and the position coordinates of the point $P_0$.

The path to be followed by $P_0$ is denoted as $(C)$. For the sake of simplicity, we consider a smooth simple curve defined by one of its point, the unitary tangent vector at this point, and its curvature $curv(s)$, with $s$ being the curvilinear coordinate along the curve. Morever, it is assumed that:

- $curv(s)$ is differentiable $(n+1)$ times. This is necessary for $P_0$ to be able to remain on $(C)$ without stopping, as it will later appear.
- The radius of any circle tangenting $(C)$ at two or more points, and the interior of which does not contain any point of the curve, is lowerbounded by some positive real number denoted as $r_{min}$. The set of circles' centers so defined is the Voronoi diagram associated with the curve [30]. This assumption implies in particular that $|curv(s)| \leq 1/r_{min}$, $\forall s$. For example, if $(C)$ is a straight line, then $r_{min} = +\infty$ and $curv(s) = 0$. If $(C)$ is a circle, then $r_{min}$ is the circle's radius and $curv(s) = 1/r_{min}$.

Under these assumptions, if the distance between $P_0$ and $(C)$ is smaller than $r_{min}$, there is a unique point on $(C)$, denoted as $P_{0,proj}$, such that $||P_0 P_{0,proj}||$ is equal to this distance.

Let $s$ denote the curvilinear coordinate at the point $P_{0,proj}$, and $(P_{0,proj}; \vec{t}, \vec{n})$ the Frenet frame on the curve at this point. The position of the point $P_0$ in the plane is fully characterized (parametrized) by the pair of variables $(s, y)$ where $y$ is the intensity of the vector $\overrightarrow{P_{0,proj} P_0}$, i.e.:

$$\overrightarrow{P_{0,proj} P_0} \;=\; y\,\vec{n} \tag{64}$$

Note that in the particular case where $(C)$ is a straight line, $s$ and $y$ coincide with classical Cartesian coordinates. For other curves, one of the control objectives will be to keep the coordinate $y$ smaller than $r_{min}$ all the time so as to avoid any ambiguity when using the parametrization $(s, y)$.

This parametrization has already been used in [27] for the control of a unicycle-type vehicle.

The motion of the point $P_0$ is then characterized by the time derivatives $\dot{s}$ and $\dot{y}$. Let:

- $\theta_t$ denote the angle between $\vec{i}_0$ and $\vec{t}(s)$
- $\theta = \alpha_0 - \theta_t$ the angle between the first vehicle's body and the curve's tangent vector $\vec{t}$.

When the first vehicle follows $(C)$ exactly, with a non-zero translational velocity, $\theta$ can only take values equal to $k\pi$ $(k \in Z)$. The generality of the proposed approach will not be reduced by assuming from now on that the desired value for the angle $\theta$ is zero.

It is then quite simple to derive the following relationships (see [27] also):

$$
\begin{array}{rcl}
\dot{s} & = & v_0 \; \frac{cos(\theta)}{1-curv(s)y} \\
\dot{y} & = & v_0 \; sin(\theta) \\
\dot{\theta}_t & = & curv(s) \; \dot{s} \\
 & = & v_0 \; \frac{curv(s)cos(\theta)}{1-curv(s)y}
\end{array}
\tag{65}
$$

By regrouping all previous equations, one obtains the following control system:

$$
\dot{X} = g_1(X)v_0 + g_2 u_2 \tag{66}
$$

with:

$$
X \; = \; \begin{bmatrix} s \\ y \\ \theta \\ \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \qquad dim(X) = n + 4
$$

$$
\begin{array}{rcl}
v_0 & = & r\omega \; \prod_{i=4}^{i=n+4} cos(x_i) \\
u_2 & = & \dot{x}_{n+4}
\end{array}
$$

$$
\begin{array}{lcl}
g_{1,1}(X) & = & \frac{cos(x_3)}{1-curv(x_1)x_2} \\
g_{1,2}(X) & = & sin(x_3) \\
g_{1,3}(X) & = & \frac{tan(x_4)}{l_1} - \frac{curv(x_1)cos(x_3)}{1-curv(x_1)x_2} \\
g_{1,4}(X) & = & \frac{1}{cos(x_4)}(\frac{tan(x_5)}{l_2} - \frac{sin(x_4)}{l_1}) \\
\vdots & & \\
g_{1,j}(X) & = & \frac{1}{\prod_{l=4}^{l=j} cos(x_j)}(\frac{tan(x_{j+1})}{l_{j-2}} - \frac{sin(x_j)}{l_{j-3}}) \qquad (5 \leq j \leq n+3) \\
\vdots & & \\
g_{1,n+4}(X) & = & 0
\end{array}
$$

$$
g_2 \; = \; \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
$$

This control system characterizes our mechanical system, as long as $X$ belongs to the set:

$$
\Omega \; = \; R\times] - r_{min}, +r_{min}[\times R \times (] - \frac{\pi}{2}, +\frac{\pi}{2}[)^{n+1} \tag{67}
$$

Although controllability of the mechanical system in $R^3 \times SO_1^{n+1}$ (i.e. the possibility of steering the system between any two configurations in finite time) has been theoretically established (see [13], for example), the control design and analysis is hereafter limited to the the set $\Omega$. In particular, the angles $\alpha_i$ $(1 \leq i \leq n+1)$ are bound to stay in the interval $]-\pi/2, +\pi/2[$. For most practical purposes, this seems to be little restrictive. Moreover, undesirable "jack-knife" effects will systematically be avoided in this way.

## 3.2 About path planning

As already pointed out, the angle $\theta$ must be equal to zero (or $\pi$) when $P_0$ moves along the desired path. This also means that the time-derivative of this angle must be equal to zero. Thus, according to the third equation of the above system, and since $y$ is also equal to zero along the path, one must have:

$$tan(\alpha_1) \; = \; \begin{cases} +l_1 curv(s) & if \; \theta = 0 \\ -l_1 curv(s) & if \; \theta = \pi \end{cases} \tag{68}$$

By taking the time derivative of this last equation, and comparing the result with the fourth system's equation, one obtains after simple calculations:

$$tan(\alpha_2) \; = \; \begin{cases} \frac{l_2}{l_1^2 curv(s)^2+1}(curv(s) + \frac{l_1 curv^{(1)}(s)}{l_1^2 curv(s)^2+1}) & if \; \theta = 0 \\ \frac{l_2}{l_1^2 curv(s)^2+1}(curv(s) - \frac{l_1 curv^{(1)}(s)}{l_1^2 curv(s)^2+1}) & if \; \theta = \pi \end{cases} \tag{69}$$

where $curv^{(1)}(s)$ is the first derivative (with respect to the curvilinear coordinate) of the path's curvature.

Repeating the above procedure $(n-1)$ times, one would obtain that, along the desired path and for each of the two possible values of $\theta$, the angles $\alpha_i$ $(1 \leq i \leq n+1)$ are functions of the path's curvature and its successive derivatives up to the order $(i-1)$. Moreover, one may verify that the correspondance from $(]-\pi/2, +\pi/2[)^{n+1}$ onto $R^{n+1}$ between the set of angles $\{\alpha_i\}_{1 \leq i \leq n+1}$ and the set $\{curv^{(i)}(s)\}_{0 \leq i \leq n}$ is **one-to-one**.

A direct consequence of this fact is that the problem of steering the system between any two configurations (satisfying the aforementionned condition imposed on the range of the angles $\alpha_i$) can be addressed as a **purely geometrical problem** consisting of finding a planar path of class $C^{n+3}$ which connects two given points in the plane (corresponding to the initial and final position of the point $P_0$), with given tangents at these points (corresponding to initial and final values of $\alpha_0$), and conditions imposed on the curvature and its $n$ successive derivatives at both extremities of the path (corresponding to initial and final values of angles $\alpha_i$ $(1 \leq i \leq n+1)$). This result has also been pointed out in [19], where it is presented as a consequence of the system's *flatness*.

Since such a path obviously exists, one finds again in this way that the system is controllable in $\Omega$.

It may also be noted that there is an abundant literature dealing with this type of geometrical problem. Solutions proposed in this domain, such as widely used Bezier's polynomial curves (splines) for example [2], could thus be of interest for people working on mobile robot path planning problems and yield methods complementary to those already explored.

### 3.3 Smooth feedback controls for path following and stabilization about a fixed desired configuration

In order to apply the results of Section 2, the system 66 should first be converted into the chained form 1, or, equivalently, to the skew-symmetric chained form 13.

We first rewrite the original system as follows:

$$\dot{X} = f_1(X)u_1 + g_2\dot{x}_{n+4} \tag{70}$$

with:

$$u_1 = v_0 \frac{cos(x_3)}{1 - curv(x_1)x_2}$$

$$\begin{array}{lcl}
f_{1,1} & = & 1 \\
f_{1,2}(X) & = & \frac{1-curv(x_1)x_2}{cos(x_3)}\ g_{1,2}(x_3) \\
f_{1,3}(X) & = & \frac{1-curv(x_1)x_2}{cos(x_3)}\ g_{1,3}(x_1, x_2, x_3, x_4) \\
f_{1,4}(X) & = & \frac{1-curv(x_1)x_2}{cos(x_3)}\ g_{1,4}(x_4, x_5) \\
 & \vdots & \\
f_{1,n+3}(X) & = & \frac{1-curv(x_1)x_2}{cos(x_3)}\ g_{1,n+3}(x_4, \ldots, x_{n+4}) \\
f_{1,n+4} & = & 0
\end{array}$$

This control system is equivalent to the original one within the reduced set:

$$\Omega_{reduced} = R\times] - r_{min}, +r_{min}[\times(] - \tfrac{\pi}{2}, +\tfrac{\pi}{2}[)^{n+2} \quad (\subset \Omega)$$

In particular, due to the choice of the input $u_1$, the variable $x_3$ (i.e. the orientation error $\theta$) has to be kept in the interval $] - \frac{\pi}{2}, +\frac{\pi}{2}[$.

This control system is directly converted into a skew-symmetric chained form via the change of coordinates $\phi_2$: $X \longmapsto Z$, with:

$$
\begin{array}{lcl}
z_1 & = & x_1 \\
z_2 & = & h_y(x_2) \\
z_3 & = & f_{1,2}(x_1, x_2, x_3)\frac{\partial h_y}{\partial x_2} \\
z_4 & = & k_1 z_2 + L_{f_1} z_3 \\
 & \vdots & \\
z_{j+3} & = & k_j z_{j+1} + L_{f_1} z_{j+2} \qquad (j \leq 2 \leq n+1) \\
 & \vdots &
\end{array}
\tag{71}
$$

where $h_y(x_2)$ is a smooth monotonic function which maps $]-r_{min}, +r_{min}[$ onto $R$, with first derivative strictly larger than a positive real number, and such that $h_y(0) = 0$. For example, $h_y(x_2) = \frac{2r_{min}}{\pi} tan(\frac{\pi y}{2r_{min}})$ is a possible candidate when $r_{min} < +\infty$. If $r_{min} = +\infty$, the simplest choice is $h_y(x_2) = x_2$.

This function is here introduced to force $|y(t)|$ $(= |x_2(t)|)$ to remain smaller than $r_{min}$ in the subsequent control analysis.

One can verify that the Jacobian matrix $\frac{\partial \phi_2}{\partial X}$ is a lower triangular matrix with non-zero components on the diagonal equal to: 1, $\frac{\partial h_y}{\partial x_2}$, $\frac{1-curv(x_1)x_2}{cos(x_3)^2}\frac{\partial h_y}{\partial x_2}$, $\frac{(1-curv(x_1)x_2)^2}{l_1 cos(x_3)^3 cos(x_4)^2}\frac{\partial h_y}{\partial x_2}$, ..., $\frac{(1-curv(x_1)x_2)^{n+1}}{(\prod_{i=1}^{i=n+1} l_j)(\prod_{j=3}^{j=n+4} cos(x_j)^{n+6-j})}\frac{\partial h_y}{\partial x_2}$.

All other components are well defined in $\Omega_{reduced}$. This matrix is thus defined and nonsingular on $\Omega_{reduced}$.

One can also easily verify that: $\phi_2(\Omega_{reduced}) = R^{n+4}$. Thus, according to the theorem 0.5 in [21] (page 13), $\phi_2$ induces a diffeomorphism of class $C^{n+1}$ ($n+1$ being the degree of differentiability of $curv(x_1)$) between $\Omega_{reduced}$ and $R^{n+4}$.

Then, by using the fact that:

$$
L_{g_2} L_{f_1}^j z_{i+3} \; = \; 0 \qquad (0 \leq i \leq n, \; 0 \leq j \leq n-i) \tag{72}
$$

and:

$$
L_{g_2} L_{f_1}^{n+1} z_3 \; = \; \frac{(1 - curv(x_1)x_2)^{n+1}}{(\prod_{i=1}^{i=n+1} l_j)(\prod_{j=3}^{j=n+4} cos(x_j)^{n+6-j})}\frac{\partial h}{\partial x_2}(x_2) \tag{73}
$$

it is straightforward to verify that the control system, with $Z$ as state vector, has the form 13, with the auxiliary control input $w_2$ defined by:

$$
w_2 \; = \; (k_{n+2} z_{n+3} + L_{f_1} z_{n+4})u_1 \; + \; (L_{g_2} L_{f_1}^{n+1} z_3)u_2 \tag{74}
$$

Once the system has been put into the skew-symmetric chained form, one can apply Proposition 2.1 to determine a control input $w_2$ which stabilizes the point $Z_2 = 0$ and thus make the mechanical system follow the path $(C)$. One can also apply Proposition 2.2 to determine smooth time-varying feedbaks which make the system converge to a given configuration on the path.

However, the stability results are only *local* in this case since the state vector $X$ must belong to $\Omega_{reduced}$. This implies that the angles $\theta$ and $\alpha_i$ $(1 \leq n+1)$ must have initial values in $]-\frac{\pi}{2}, +\frac{\pi}{2}[$, and that the initial distance $|y(0)|$ must be smaller than $r_{min}$.

**Remark:** The method described above allows to asymptotically stabilize the mechanical system about any configuration such that $|\alpha_i| < \frac{\pi}{2}$ $(1 \leq n+1)$. In other studies ([20], [29], for example), only configurations with zero angles $\alpha_i$ (all trailers aligned) have been considered.

**Example of the car with one trailer**:

In this example, the path to be followed by the system is chosen as a straight line ($\Rightarrow curv(s) = 0, \ \forall s$), in order to simplify the calculations and avoid too cumbersome notations.

The system's equations are:

$$\begin{array}{rcl} \dot{s} & = & v_0 cos(\theta) \\ \dot{y} & = & v_0 sin(\theta) \\ \dot{\theta} & = & v_0 \frac{tan(\alpha_1)}{l_1} \\ \dot{\alpha}_1 & = & v_0 \frac{1}{cos(\alpha_1)}(\frac{tan(\alpha_2)}{l_2} - \frac{sin(\alpha_1)}{l_1}) \\ \dot{\alpha}_2 & = & u_2 \end{array} \tag{75}$$

with: $v_0 \ = \ r\omega cos(\alpha_1) cos(\alpha_2)$

They can be written in the form 70, with:

$$f_1 = \left[ \begin{array}{c} 1 \\ tan(\theta) \\ \frac{tan(\alpha_1)}{l_1 cos(\theta)} \\ \frac{1}{cos(\alpha_1)cos(\theta)}(\frac{tan(\alpha_2)}{l_2} - \frac{sin(\alpha_1)}{l_1}) \\ 0 \end{array} \right] \qquad g_2 = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right] \tag{76}$$

and:

$$u_1 \ = \ r\omega cos(\theta) cos(\alpha_1) cos(\alpha_2) \tag{77}$$

The $Z$ coordinates, which transform the system into the skew-symmetric chained form, are (with $h_y(x_2) = x_2$):

$$
\begin{array}{lllll}
z_1 & = & s & & \\
z_2 & = & y & & \\
z_3 & = & tan(\theta) & & \\
z_4 & = & k_1 z_2 + L_{f_1} z_3 & = & k_1 y + \frac{tan(\alpha_1)}{l_1 cos(\theta)^3} \\
z_5 & = & k_2 z_3 + L_{f_1} z_4 & = & (k_1+k_2) tan(\theta) + \frac{3 sin(\theta) tan(\alpha_1)^2}{l_1^2 cos(\theta)^5} + \frac{1}{l_1 cos(\alpha_1)^3 cos(\theta)^4} \left( \frac{tan(\alpha_2)}{l_2} - \frac{sin(\alpha_1)}{l_1} \right)
\end{array}
\tag{78}
$$

The new second control input is:

$$
\begin{array}{lll}
w_2 & = & [\ k_3 z_4 + L_{f_1} z_5) u_1 \ + \ (L_{g_2} L_{f_1}^2 z_3) u_2 \\
& = & [(k_1 k_3 y + (k_1+k_2+k_3) \frac{tan(\alpha_1)}{l_1 cos(\theta)^3} + (1 + 4 sin(\theta)^2) \frac{3 tan(\alpha_1)^3}{l_1^3 cos(\theta)^7} \\
& & + \frac{10 sin(\theta) tan(\alpha_1) - cos(\theta)}{l_1^2 cos(\theta)^6 cos(\alpha_1)^3} \left( \frac{tan(\alpha_2)}{l_2} - \frac{sin(\alpha_1)}{l_1} \right) + \frac{3 sin(\alpha_1)}{l_1 cos(\theta)^5 cos(\alpha_1)^5} \left( \frac{tan(\alpha_2)}{l_2} - \frac{sin(\alpha_1)}{l_1} \right)^2 \ ] \ u_1 \\
& & + \ \frac{1}{l_1 l_2 cos(\theta)^4 cos(\alpha_1)^3 cos(\alpha_2)^2} \ u_2
\end{array}
\tag{79}
$$

and a control suggested by Proposition 2.1 is:

$$
\begin{array}{lll}
w_2 & = & -k'_{w_2} |u_1| \ z_5 \qquad (k'_{w_2} > 0) \\
& = & -k'_{w_2} |u_1| \ [\ (k_1+k_2) tan(\theta) + \frac{3 sin(\theta) tan(\alpha_1)^2}{l_1^2 cos(\theta)^5} + \frac{1}{l_1 cos(\alpha_1)^3 cos(\theta)^4} \left( \frac{tan(\alpha_2)}{l_2} - \frac{sin(\alpha_1)}{l_1} \right) \ ]
\end{array}
\tag{80}
$$

The control input $u_2$ is then obtained by eliminating $w_2$ between the two previous equations.

This control can be used to follow the desired path (a straight line, here).

In order to stabilize the system about a given absciss $s_d$ along the path, Proposition 2.2 suggests complementing the above control with a time-varying feedback $u_1$, such as:

$$
u_1 \ = \ -k_{u_1}(s - s_d) \ + \ h(y, \theta, \alpha_1, \alpha_2, t) \qquad (k_{u_1} > 0) \tag{81}
$$

with $h(y, \theta, \alpha_1, \alpha_2, t)$ satisfying the conditions given in the Proposition.

### 3.4 A modification to broaden the stability domain

A practical shortcoming of the above control design method is the necessity of starting with an orientation error $|\theta|$ smaller than $\pi/2$.

In this section, it is shown how this limitation can be removed by considering a more global change of coordinates which converts the initial control system to a modified chained form.

The results given in this section are direct extensions of those given in [27] in the particular case of a single unicycle-type vehicle. Simulation results can be found in the same reference.

The new transformation $\phi_3 : \ X \longmapsto Z$ that is considered is the following:

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= h_y(x_2) \\
z_3 &= x_3 \\
z_4 &= k_1 \frac{sin(x_3)}{x_3} \frac{\partial h_y}{\partial x_2}(x_2) z_2 \ + \ g_{1,3}(x_1, x_2, x_3, x_4) \\
z_5 &= k_2 z_3 \ + \ L_{g_1} z_4 \\
&\vdots \\
z_{j+4} &= k_{j+1} z_{j+2} \ + \ L_{g_1} z_{j+3} \qquad (2 \leq j \leq n) \\
&\vdots
\end{aligned}
\tag{82}
$$

where:

- $g_1(X)$ is the vector field involved in the system's representation 66
- $k_j \ (1 \leq j \leq n+1)$ are positive real numbers
- $h_y(x_2)$ is the monotonic function introduced before

**Remark:** Instead of $z_3 = x_3$, one may also take $z_3 = h_\theta(x_3)$, with $h_\theta(x_3)$ being a smooth monotonic function, alike $h_y(x_2)$, which maps an open interval containing $]-\pi, +\pi[$ into $R$. For example, $h_\theta(x_3) = tan(kx_3)$ with $k < \frac{1}{2}$ can be used. In this case the coordinate $z_4$ becomes: $z_4 = k_1 \frac{sin(x_3)}{h_\theta(x_3)} \frac{\partial h_y}{\partial x_2}(x_2) z_2 + \ g_{1,3}(x_1, x_2, x_3, x_4)$, and all subsequent coordinates are modified accordingly.

One can verify that $\phi_3(\Omega) = R^{n+4}$ and that the Jacobian matrix $\frac{\partial \phi_3}{\partial X}(X)$ is a lower-triangular matrix with non-zero components on the diagonal equal to: $1$, $\frac{\partial h_y}{\partial x_2}(x_2)$, $1$, $\frac{1}{l_1 cos(x_4)^2}$, $\frac{1}{l_1 l_2 cos(x_4)^3 cos(x_5)^2}$, $\ldots$, $\frac{1}{(\prod_{i=1}^{i=n+1} l_i)(\prod_{j=4}^{j=n+4} cos(x_j)^{n+6-j})}$.

The change of coordinates $\phi_3$ thus induces a diffeomorphism of class $C^{n+1}$ between $\Omega$ and $R^{n+4}$.

Then, by using:

$$
L_{g_2} L_{g_1}^j z_{i+3} \ = \ 0 \qquad (0 \leq i \leq n, \ 0 \leq j \leq n-i) \tag{83}
$$

$$
L_{g_2} L_{g_1}^n z_4 \ = \ \frac{1}{(\prod_{i=1}^{i=n+1} l_j)(\prod_{j=4}^{j=n+4} cos(x_j)^{n+6-j})} \tag{84}
$$

with the following auxiliary control input $w_2$:

$$
w_2 \ = \ (k_{n+2} z_{n+3} + L_{g_1} z_{n+4}) v_0 \ + \ (L_{g_2} L_{g_1}^n z_4) u_2 \tag{85}
$$

it is simple to verify that the control system, expressed in terms of the new coordinates, has the following skew-symmetric chained form:

$$
\begin{array}{lcl}
\dot{z}_1 & = & v_0 \frac{cos(z_3)}{1-curv(z_1)x_2} \\
\dot{z}_2 & = & v_0 \frac{sin(z_3)}{z_3} \frac{\partial h_y}{\partial x_2}(x_2) z_3 \\
\dot{z}_3 & = & -k_1 v_0 \frac{sin(z_3)}{z_3} \frac{\partial h_y}{\partial x_2}(x_2) z_2 \ + \ v_0 z_4 \\
\dot{z}_4 & = & -k_2 v_0 z_3 \ + \ v_0 z_5 \\
 & \vdots & \\
\dot{z}_{j+4} & = & -k_{j+2} v_0 z_{j+3} \ + \ v_0 z_{j+5} \qquad (1 \leq j \leq n-1) \\
 & \vdots & \\
\dot{z}_{n+4} & = & -k_{n+2} v_0 z_{n+3} \ + \ w_2
\end{array}
\tag{86}
$$

with: $x_2 = h_y^{-1}(z_2)$, and $v_0 = r\omega \prod_{i=4}^{i=n+4} cos(x_i)$.

Although this system is not exactly the same as the skew-symmetric chained system 13, a result very similar to Proposition 2.1 can be derived. More precisely, we have:

**Proposition 3.1**
*If $|v_0(t)|$ and $|\dot{v}_0(t)|$ are bounded, and if the control:*

$$
w_2 \ = \ -k_{w_2}(v_0)\, z_{n+4} \quad (k_{w_2}(.) : \textit{ continuous application strictly positive on } R - \{0\})
\tag{87}
$$

*is applied to the system 86, then the positive function:*

$$
V(Z_2) = 1/2 \ ( \ z_2^2 + (1/k_1) z_3^2 + (1/k_1 k_2) z_4^2 + \cdots + (1/ \prod_{j=1}^{j=n+2} k_j) z_{n+4}^2 \ )
\tag{88}
$$

*is non-increasing along the system's solutions, and thus asymptotically converges to some limit value $V_{lim}$ (which* a priori *depends on the initial conditions).*

*Moreover, $v_0(t)V(Z_2(t))$ asymptotically converges to zero.*

*Therefore, if $v_0(t)$ does not to zero, then $V_{lim} = 0$. The point $Z_2 = 0$ is thus globally asymptotically stabilized in this case.*

**Proof of Proposition 3.1**:
The proof is quite similar to the proof of Proposition 2.1 except that one has to show at some point that the convergence of $v_0 \frac{sin(z_3)}{z_3} \frac{\partial h_y}{\partial x_2}(x_2) z_2$ and $v_0 z_3$ to zero yields the convergence of $v_0 z_2$ to zero.

Since $|z_2(t)|$ is bounded (from the boundedness of the Lyapunov function), $\frac{\partial h_y}{\partial x_2}(x_2(t))$ is also upperbounded, and $v_0 z_3 \frac{\partial h_y}{\partial x_2}(x_2) z_2$ thus tends to zero. Therefore, ${v_0}^2 ((\frac{sin(z_3)}{z_3})^2 + {z_3}^2)(\frac{\partial h_y}{\partial x_2}(x_2))^2 {z_2}^2$ also tends to zero.

By assumption, $\frac{\partial h_y}{\partial x_2}(x_2)$ is bounded from below by a positive real number. Moreover, the function $(\frac{sin(z_3)}{z_3})^2 + {z_3}^2$ is itself strictly larger than some positive real number. Along a system's solution, it is also bounded from above, since $|z_3(t)|$ is bounded. Using these bounds in the previous convergence result, it is found that ${v_0}^2{z_2}^2$ tends to zero.
$\diamond$

The remarks made after Proposition 2.1 also hold in this case. In particular, adequate values for the control "gains" $k_j$ $(1 \leq j \leq n+2)$ can be determined by comparing, in the neighbourhood of $Z_2 = 0$, the control $u_2$ provided by the Proposition and relation 85, with the linearizing feedback 7. Since these gains do not *a priori* depend on the path's shape, one may also use, for this comparison, a simpler linear control calculated from the system's tangent linear approximation about the equilibrium $(Z_2 = 0, u_2 = 0)$, assuming that the path to be followed is a straight line and that the velocity $v_0$ is constant.

The above theorem provides us with solutions to the path following problem. The problem of stabilizing the system about a fixed desired configuration requires asymptotical convergence of the full state vector $Z$ to zero. A smooth time-varying feedback solution is given in the next complementary Proposition.

**Proposition 3.2**

*Consider the same control as in Proposition 3.1:*

$$w_2 \; = \; -k_{w_2}(v_0)\; z_{n+4} \tag{89}$$

*complemented with the following time-varying control:*

$$v_0 \; = \; -k_{v_0} h_s(z_1) \; + \; h(Z_2, t) \tag{90}$$

*where:*

- $k_{v_0}$ *is a positive real number.*
- $h_s(.)$ *is a function of class* $C^2$ *which maps* $R$ *into a* bounded *interval of* $R$*, and such that: i)* $h_s(0) = 0$*, ii)* $0 < h_s^{(1)}(x) < +\infty$*,* $\forall x$*, and iii)* $|h_s^{(2)}(x)| < +\infty$*,* $\forall x$*.*

  *Take, for example, the sigmoid function:* $h_s(x) = \frac{exp(ax)-1}{exp(ax)+1}$ *(*$a > 0$*).*
- $h(Z_2, t)$ *is a function with the same properties as in Proposition 2.2.*

*This control globally asymptotically stabilizes the origin* $Z = 0$ *of the system 86.*

**Proof of theorem 3.2**:

The first part of the proof consists in showing that $v_0(t)$ and its time derivative are bounded along any system's solution.

Since $||Z_2(t)||$ is bounded (due to the boundedness of the Lyapunov function considered in Theorem 3.1), it is clear, from the expression of the control $v_0$ and the properties of the functions $h_s(z_1)$ and $h(Z_2, t)$, that $|v_0(t)|$ is bounded. As a consequence, $||\dot{Z}(t)||$ is also bounded.

Taking the time derivative of the $v_0$ control law expression:

$$\dot{v}_0 \;=\; [-k_{v_0} h_s^{(1)}, \frac{\partial h}{\partial Z_2}]\dot{Z} + \frac{\partial h}{\partial t} \tag{91}$$

which, in view of the boundedeness of $||Z_2(t)||$ and $||\dot{Z}(t)||$, implies that $|\dot{v}_0(t)|$ is bounded.

Although $v_0$ is not, strictly speaking, a function of time only (since it is a feedback control), it can viewed as such along any system's solution, and the results of Proposition 3.1 do apply.

In particular, if $v_0(t)$ does not tend to zero, then $||Z_2(t)||$ tends to zero. In this case, $h(Z_2(t), t)$ tends to zero (from condition $C_1$ and by uniform continuity). From the first system's equation, we also have:

$$\dot{z}_1(t) \;=\; -k_{v_0} h_s(z_1(t)) \;+\; o(t) \qquad with \;\lim_{t\to\infty} o(t) = 0 \tag{92}$$

Using the properties of the function $h_s(.)$, this equation implies that the following proposition is true:

$$\forall \epsilon > 0,\; \exists \eta > 0,\; \exists t_0 \;:\; (t > t_0 \;and\; |z_1(t)| \geq \epsilon) \Rightarrow (z_1(t)\dot{z}_1(t) < -\eta) \tag{93}$$

Since $z_1^2(t)$ cannot remain larger than $\epsilon^2$ with a negative derivative smaller than $-2\eta$, there is a time $t_1$ ($\geq t_0$) such that $|z_1(t_1)| < \epsilon$. Moreover, after the time $t_1$, $|z_1(t)|$ remains smaller than $\epsilon$ (since $\epsilon^2$ cannot be reached from below by $z_1^2(t)$ with a negative derivative). The above proposition thus implies:

$$\forall \epsilon > 0,\; \exists t_1 \;:\; (t > t_1) \Rightarrow (|z_1(t)| < \epsilon) \tag{94}$$

This is a characterization of the convergence of $z_1(t)$ to zero. In view of the expression of $v_0$, this in turn implies that $v_0(t)$ tends to zero (contradiction).

Therefore $v_0(t)$ must tend to zero, implying in turn that $w_2(t)$ and $\dot{Z}(t)$ tend to zero. Now, in view of 91:

$$\dot{v}_0(t) \;=\; \frac{\partial h}{\partial t}(Z_2(t), t) + o(t)$$

Since $\frac{\partial h}{\partial t}(Z_2(t), t)$ is uniformly continuous (its time derivative is bounded), $\dot{v}_0(t)$ tends to zero (Barbalat's lemma). Therefore, $\frac{\partial h}{\partial t}(Z_2(t), t)$ also tends to zero.

From there, the proof goes on like the proof of Proposition 2.2.

$\diamond$.

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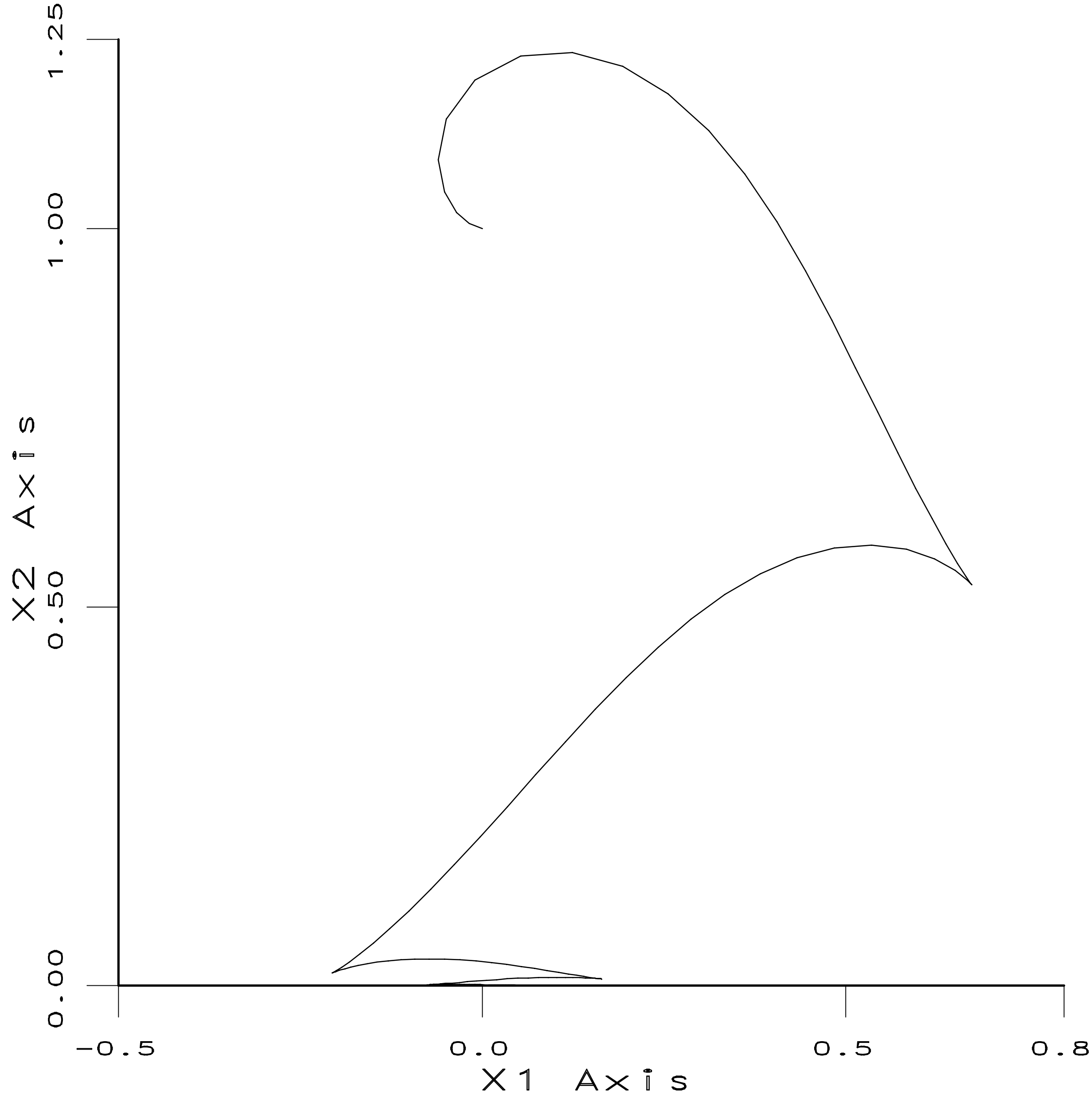
Fig. 1.a : motion in x-y plane



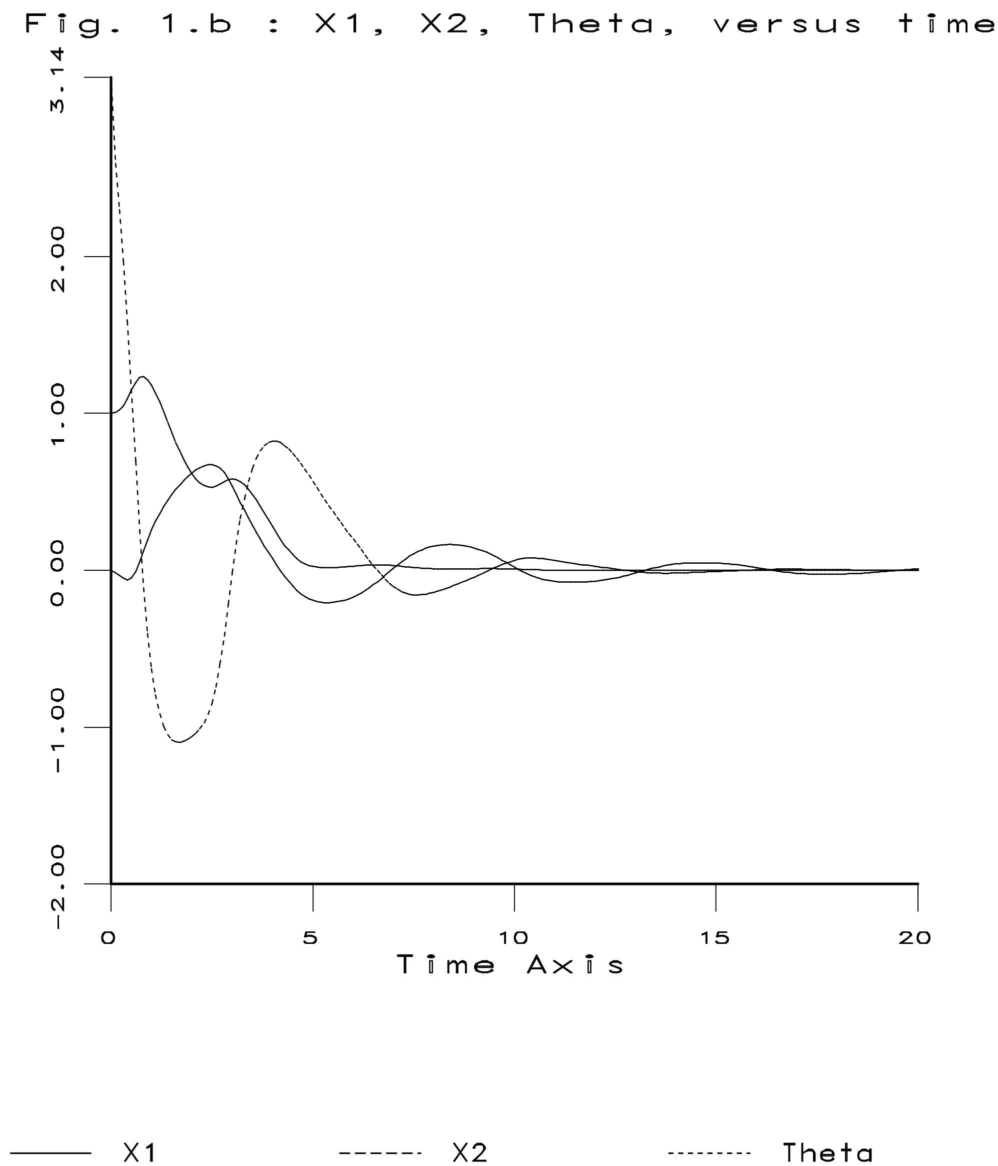


Fig. 1.b : X1, X2, Theta, versus time
3.14
2.00
1.00
0.00
-1.00
-2.00
0
5
10
15
20
Time Axis
X1
X2
Theta

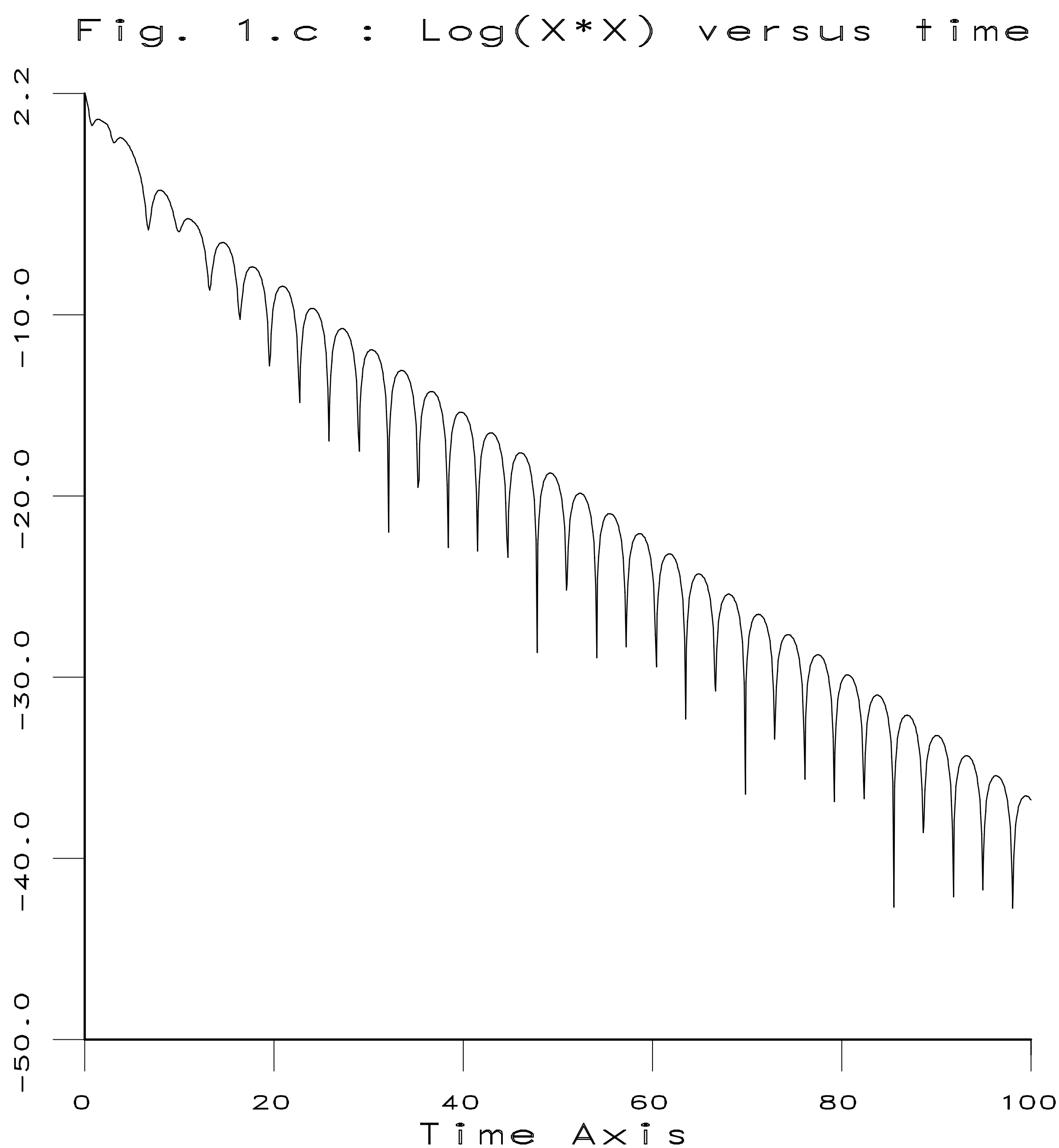


Fig. 1.c : Log(X*X) versus time
2.2
-10.0
-20.0
-30.0
-40.0
-50.0
0
20
40
60
80
100
Time Axis

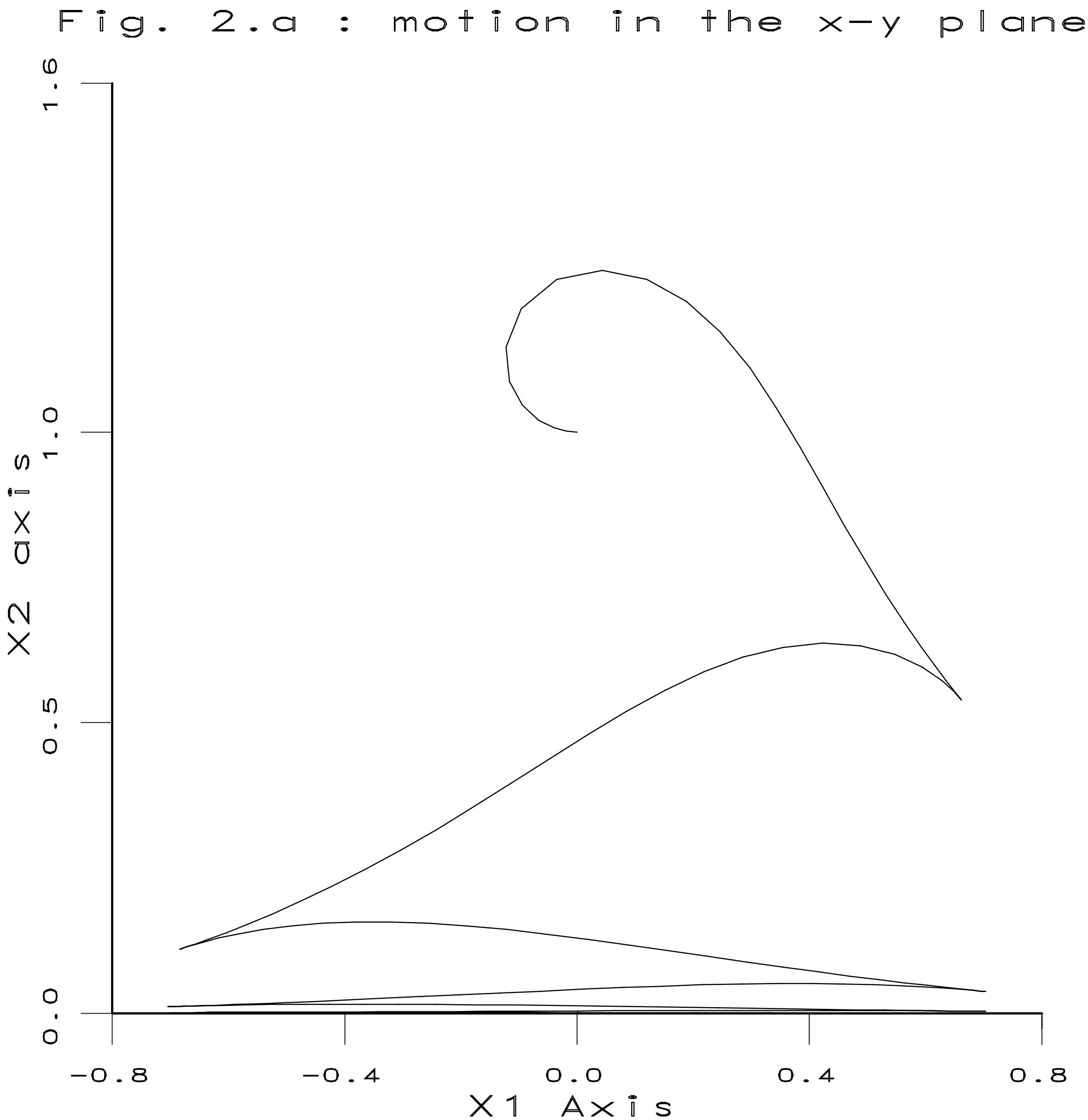
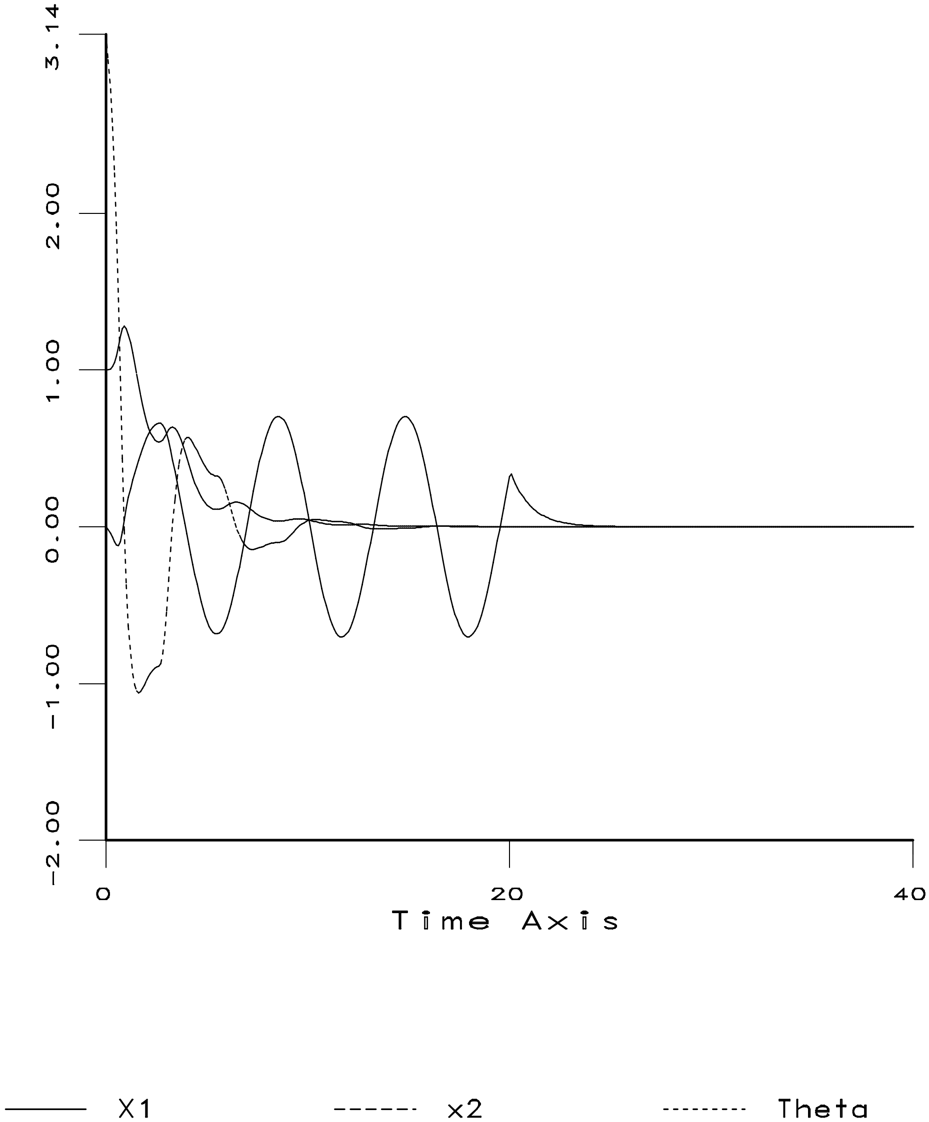


Fig. 2.a : motion in the x-y plane
X2 axis
1.6
1.0
0.5
0.0
-0.8
-0.4
0.0
0.4
0.8
X1 Axis

Fig. 2.b : X1, X2, Theta, versus time



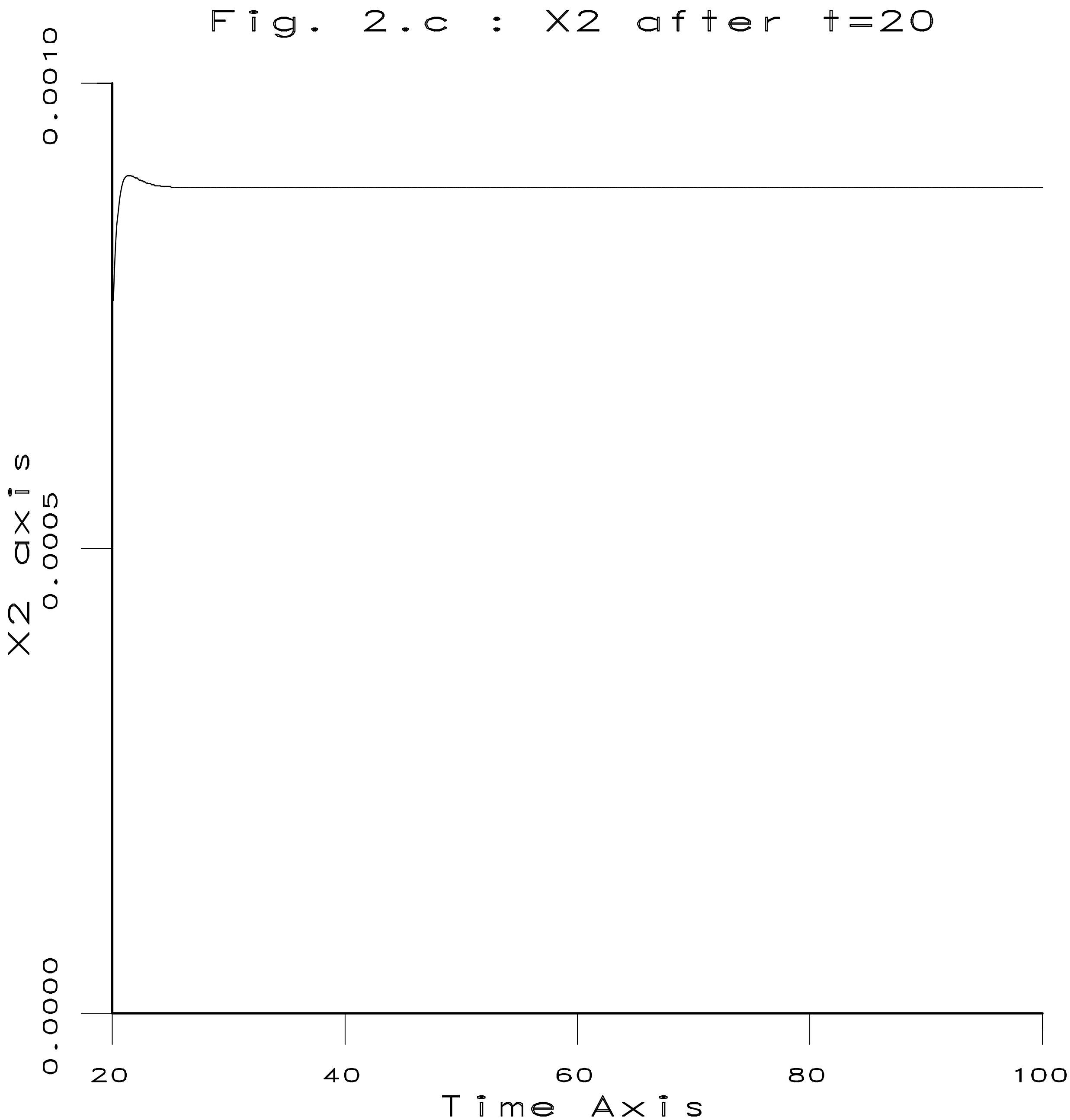


Fig. 2.c : X2 after t=20
0.0010
0.0005
0.0000
X2 axis
20
40
60
80
100
Time Axis

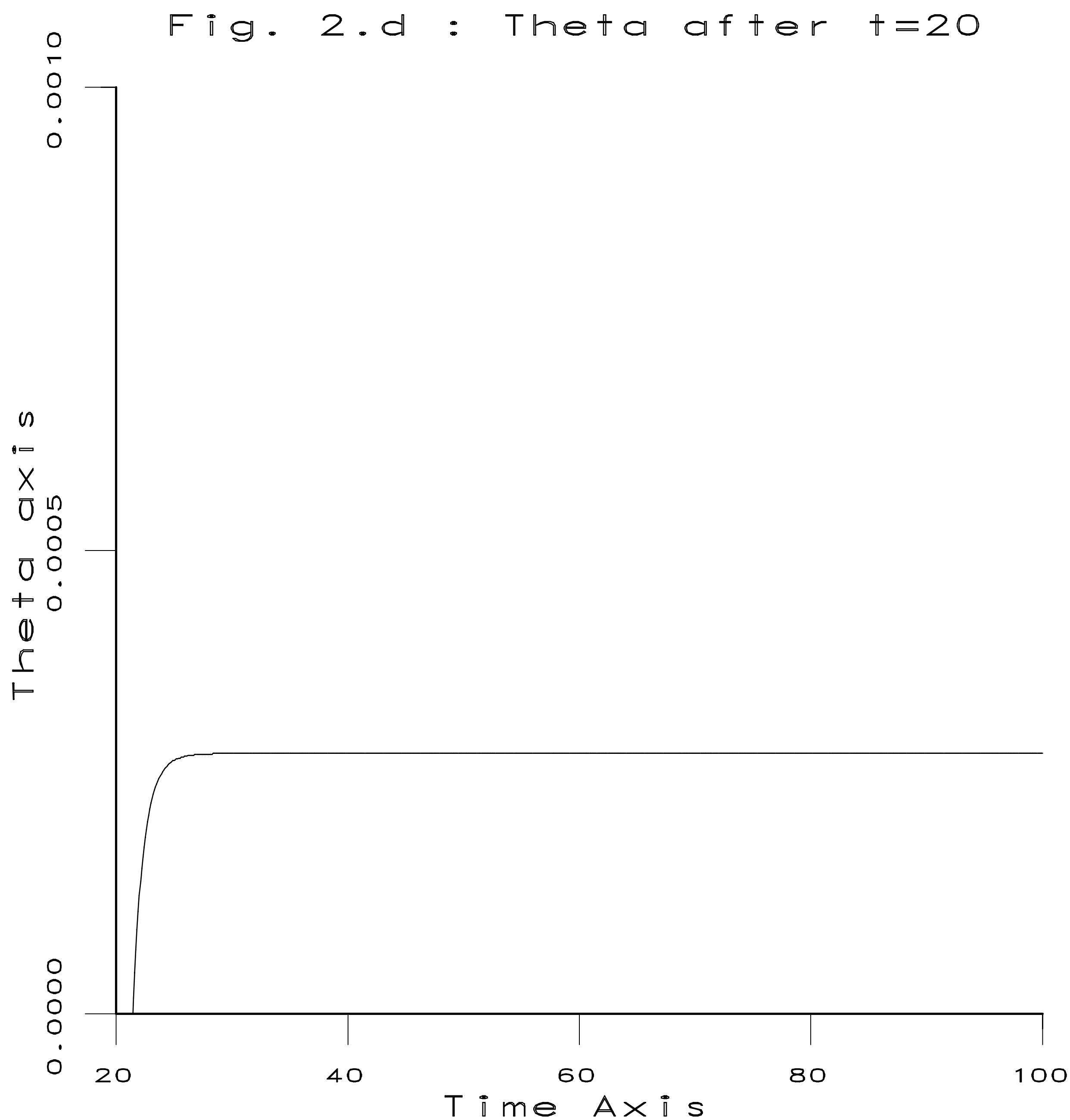


Fig. 2.d : Theta after t=20
0.0010
0.0005
0.0000
Theta axis
20
40
60
80
100
Time Axis

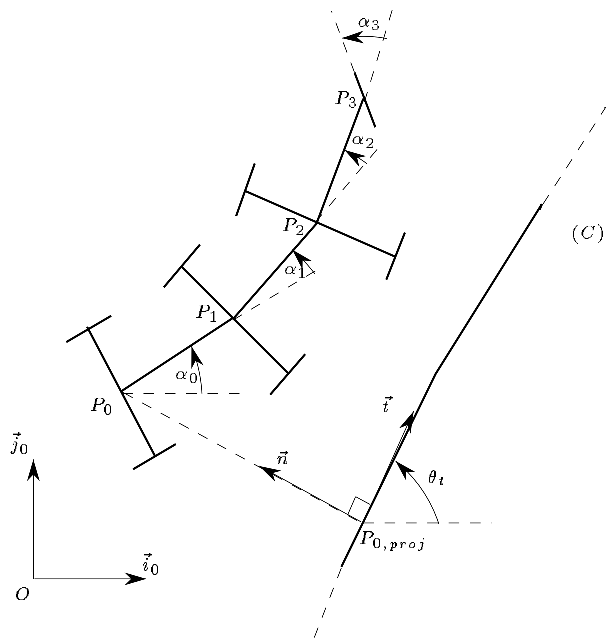



**Fig. 3:** car with two trailers